Pure Mathematics Curriculum and Assessment Guide (Advanced Level)
The Summary of Changes to the Contents of Syllabuses for
Secondary Schools-AL Applied Mathematics (1992)

<table>
<thead>
<tr>
<th>Unit</th>
<th>Concept</th>
<th>Sub-unit</th>
<th>Content deleted</th>
<th>Time Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Vectors</td>
<td>1.9</td>
<td>The whole sub-unit &quot;Triple Product&quot; is deleted</td>
<td>2</td>
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<tr>
<td></td>
<td></td>
<td>1.12</td>
<td>&quot;The moment of a force in vector form about a line in $\mathbb{R}^3$&quot; is deleted</td>
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<tr>
<td>9</td>
<td>Simple Harmonic Motion</td>
<td>9.3</td>
<td>The whole sub-unit &quot;Forced Oscillation&quot; is deleted</td>
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<tr>
<td>11</td>
<td>Motion of a Rigid Body</td>
<td>11.4</td>
<td>The whole sub-unit &quot;General Motion of a Rigid Body&quot; is deleted</td>
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<tr>
<td>13</td>
<td>Second Order Differential Equations and its Applications</td>
<td>13.7</td>
<td>On the 2nd line of Point 1, the words &quot;and forced oscillation&quot; is deleted</td>
<td>0</td>
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</tbody>
</table>

Total 24

Percentage (out of 372) 6.5

Mathematics Education Section
Education and Manpower Bureau
April 2004
Revised Sixth Form Mathematics Curriculum

1. A revised curriculum for Additional Mathematics has been implemented at S4 since September 2002 and will be first examined in the HKCEE in 2004. Changes in the Additional Mathematics curriculum might have an impact on the learning and teaching of the sixth form mathematics subjects.

2. The CDC Committee on Mathematics Education recommends that the ASL Mathematics & Statistics and ASL Applied Mathematics curricula would remain unchanged, while the curriculum contents of the AL Pure Mathematics and the AL Applied Mathematics should be trimmed down. The purpose is to allow students more time and space to develop their thinking abilities, and to better cover the two AL courses over a period of two years.

3. The two revised AL curricula will be implemented at S6 in September 2004 and first examined in Hong Kong Advanced Level Examination in 2006.
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## Chapter

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   - Objectives of the AL Pure Mathematics Curriculum

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Membership of the CDC-HKEAA Joint Working Party on Revision of Sixth Form Mathematics Curriculum

The membership since 21 March 2002 has been as follows:

Chairman

Chief Curriculum Development Officer (Mathematics)
Education and Manpower Bureau
(Mr KWAN Siu-kam)
(to 5 May 2003)

Members

Mr CHAN Chi-hung
Ms CHAN Ching-yee
Prof CHENG Shiu-yuen
Mr CHU Kan-kong
Mr FUNG Chi-yeung
Ms HO Hau-yee
Mr HUNG Chun-wah
Mr KWOK Ka-keung
Mr LEE Tak-fai
Mr LEUNG Kwong-shing
Mr TSUI Pee-tak
Mr WAN Tak-wing
Mr WONG Ka-lok
Dr WONG Ngai-ying
Dr WU Ka-ho
Mr WU Keung-fai
Ms YAN Pui-lung
Mr YIP Kai-to
Mr YU Kar-ming

Recorders

Curriculum Development Officer (Mathematics)
Education and Manpower Bureau
(Mr CHIANG Kin-nam)
(to 1 July 2002)

Subject Officer (Mathematics)
Hong Kong Examinations and Assessment Authority
(Mr CHU Kan-kong)
(from 2 July 2002)
Membership of the CDC-HKEAA Joint Working Group on Revision of AL Pure Mathematics

The membership since 11 April 2002 has been as follows:

**Chairman**

Senior Curriculum Development Officer (Mathematics)
Education and Manpower Bureau
(Mr LEUNG Kwong-shing)

**Members**

Mr CHAN Chi-hung
Ms CHAN Ching-yee
Prof CHENG Shiu-yuen
Mr CHU Kan-kong
Mr FUNG Chi-yeung
Ms HO Hau-yee
Mr KWAN Siu-kam
Mr KWOK Ka-keung
Dr WONG Ngai-ying
Ms YAN Pui-lung
Mr YIP Kai-to
Mr YU Kar-ming

** Recorder**

Curriculum Development Officer (Mathematics)
Education and Manpower Bureau
(Mr WAI Kwok-keung)
PREAMBLE

This Curriculum and Assessment Guide is one of the series jointly prepared by the Hong Kong Curriculum Development Council (CDC) and the Hong Kong Examinations and Assessment Authority (HKEAA). It forms the basis for learning and teaching of the subject curriculum as well as for setting public assessment. The issue of this single document on curriculum and assessment aims at conveying a clear message to the public that assessment is an integral part of the school curriculum and at promoting the culture of “assessment for learning” to improve learning and teaching.

The CDC is an advisory body giving recommendations to the Hong Kong Special Administrative Region Government on all matters relating to curriculum development for the school system from kindergarten to sixth form. Its membership includes heads of schools, practising teachers, parents, employers, academics from tertiary institutions, professionals from related fields or related bodies, representatives from the HKEAA and the Vocational Training Council, as well as officers from the Education and Manpower Bureau.

The HKEAA is an independent statutory body responsible for the conduct of the Hong Kong Certificate of Education Examination and the Hong Kong Advanced Level Examination. The governing council of the HKEAA includes members who are mainly drawn from the school sector, tertiary institutions, government bodies, professionals and persons experienced in commerce and industry.

This Curriculum and Assessment Guide is recommended by the Education and Manpower Bureau for use in secondary schools. The subject curriculum developed leads to the appropriate examination provided by the HKEAA. In this connection, the HKEAA has issued a handbook to provide information on the format of the public examination of the subject and the related rules and regulations.
The CDC and HKEAA will keep the subject curriculum under constant review and evaluation in the light of classroom experiences, students' performance in the public assessment, and the changing needs of society and students. All comments and suggestions on this Curriculum and Assessment Guide should be sent to:

Chief Curriculum Development Officer (Mathematics)
Curriculum Development Institute
Education and Manpower Bureau
4/F Kowloon Government Offices
405 Nathan Road
Yau Ma Tei
Kowloon
Chapter 1  Aims and Objectives

Mathematics pervades all aspects of life and has been central to nearly all major scientific and technological advances. Many of the developments and decisions made in our Community rely to an extent on the use of mathematics. Mathematics is considered as a powerful means of communication, a tool for studying other disciplines, an intellectual endeavour, a mode of thinking and a discipline through which students can develop their ability to appreciate the beauty of nature, think logically and make sound judgment (CDC, 2002)\(^1\). It is valuable to help students develop necessary skills for lifelong learning. Besides foundation skills and knowledge in mathematics for all citizens in the society, it is also important to widen mathematics experience to those students who are mathematically inclined.

The Advanced Level (AL) Pure Mathematics Curriculum is a two-year sixth form course designed for students intending to continue their studies in mathematics, engineering, science and technology. Students studying this curriculum are expected to have acquired mathematical knowledge at the Certificate of Education level, but previous knowledge of Additional Mathematics at the Certificate of Education level is not required.

This curriculum and assessment guide is presented as a revised edition of the Syllabuses for Secondary Schools – Pure Mathematics (Advanced Level) 1992. The curriculum has been scheduled for implementation in schools with effect from September 2004 at Secondary 6 and the first public examination will be held in 2006.

Overall Aims of Mathematics Education

The overall aims of mathematics education (CDC, 2000)\(^2\) are to develop:

- our youngsters' knowledge, skills and concepts of mathematics and to enhance their confidence and interest in mathematics, so that they can master mathematics effectively and are able to formulate and solve problems from a mathematical perspective; and


• their thinking abilities and positive attitudes towards learning mathematics and build related generic skills throughout their life time.

Objectives of the AL Pure Mathematics Curriculum

The objectives of the AL Pure Mathematics Curriculum are to:

• develop students’ understanding of more advanced mathematical concepts and processes and build up better foundations for further studies in the fields of mathematics, engineering, science and technology;

• strengthen students’ abilities to conceptualize, inquire and reason mathematically, and to use mathematics to formulate and solve problems in mathematical context and other disciplines;

• strengthen students’ abilities to communicate with others logically and critically in mathematical languages; and

• develop students a positive attitude towards mathematics learning and the capability of appreciating the aesthetic nature and cultural aspect of mathematics.
Chapter 2  Curriculum Framework

This curriculum is adapted from the Syllabuses for Secondary Schools – Pure Mathematics (Advanced Level) 1992 (referred as Syllabus 1992 hereafter). Some topics have been deleted or trimmed from the Syllabus 1992. The relevant changes and the comparison between this curriculum and the Syllabus 1992 can be found in Appendices 1 and 2 respectively. The rationale of the revision is to create curriculum space for consolidating concepts and adjusting teaching strategies (to cater for students’ individual differences), etc. so as to improve the learning of AL Pure Mathematics. The total teaching time for this curriculum should be unchanged when compared with the Syllabus 1992 to serve the said rationale (refer to the suggested time allocation on page 8).

Instead of dividing the contents of the curriculum into dimensions as in the secondary mathematics curriculum, they are divided into 2 topic areas, namely “Algebra” and “Calculus and Analytical Geometry”. “Algebra” consists of 9 units while “Calculus and Analytical Geometry” 7 units. For each unit, specific learning objectives are given to provide a sharper focus. The subject matter of each unit is broken into sub-units to facilitate learning/teaching and achievement of the specific learning objectives.

Some of the contents in the curriculum may overlap with those in the Additional Mathematics Curriculum, but it should be noted that they may have different approaches and depths of treatment. Teachers should also note that knowledge of the contents of the Additional Mathematics Curriculum is not required in studying AL Pure Mathematics.

Content and Specific Learning Objectives

<table>
<thead>
<tr>
<th>Unit</th>
<th>Content</th>
<th>Specific Learning Objectives</th>
</tr>
</thead>
<tbody>
<tr>
<td>A1</td>
<td>The Language of Mathematics</td>
<td>1. To understand the first notion of set language</td>
</tr>
<tr>
<td></td>
<td>1.1 Set Language</td>
<td>2. To understand the first notion of logic</td>
</tr>
<tr>
<td></td>
<td>1.2 Simple Logic</td>
<td></td>
</tr>
<tr>
<td>A2</td>
<td>Functions</td>
<td>1. To recognize function as a fundamental tool in other branches of mathematics</td>
</tr>
<tr>
<td></td>
<td>2.1 Functions and their graphs</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2.2 Properties and operations of</td>
<td></td>
</tr>
<tr>
<td><strong>functions</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>---------------</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>2.3 Algebraic functions</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>2.4 Trigonometric functions and their formulae</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>2.5 Exponential and logarithmic functions</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>2.</strong> To sketch and to describe the shapes of different functions</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th><strong>A3 Mathematical Induction</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>3.1</strong> The Principle of Mathematical Induction and its applications</td>
</tr>
<tr>
<td><strong>3.2</strong> Other common variations of the Principle of Mathematical Induction and their applications</td>
</tr>
<tr>
<td><strong>1.</strong> To understand the Principle of Mathematical Induction</td>
</tr>
<tr>
<td><strong>2.</strong> To apply the Principle of Mathematical Induction to prove propositions involving integers</td>
</tr>
<tr>
<td><strong>3.</strong> To be able to modify the Principle of Mathematical Induction to suit different purposes</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th><strong>A4 Inequalities</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>4.1</strong> Absolute inequalities</td>
</tr>
<tr>
<td><strong>4.2</strong> A.M. ≥ G.M.</td>
</tr>
<tr>
<td><strong>4.3</strong> Cauchy-Schwarz’s inequality</td>
</tr>
<tr>
<td><strong>4.4</strong> Conditional inequalities</td>
</tr>
<tr>
<td><strong>1.</strong> To learn the elementary properties of inequalities</td>
</tr>
<tr>
<td><strong>2.</strong> To prove simple absolute inequalities</td>
</tr>
<tr>
<td><strong>3.</strong> To solve simple conditional inequalities</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th><strong>A5 The Binomial Theorem for Positive Integral Indices</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>5.1</strong> The binomial theorem for positive integral indices</td>
</tr>
<tr>
<td><strong>5.2</strong> Applications of the binomial theorem for positive integral indices</td>
</tr>
<tr>
<td><strong>5.3</strong> Simple properties of the binomial coefficients</td>
</tr>
<tr>
<td><strong>1.</strong> To learn and apply the binomial theorem for positive integral indices</td>
</tr>
<tr>
<td><strong>2.</strong> To study the simple properties of the binomial coefficients</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th><strong>A6 Polynomials and Equations</strong></th>
</tr>
</thead>
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<td><strong>6.1</strong> Polynomials with real coefficients in one variable</td>
</tr>
<tr>
<td><strong>6.2</strong> Rational functions</td>
</tr>
<tr>
<td><strong>6.3</strong> Polynomial equations with real coefficients in one variable</td>
</tr>
<tr>
<td><strong>1.</strong> To learn the properties of polynomials with real coefficients in one variable</td>
</tr>
<tr>
<td><strong>2.</strong> To learn division algorithm, remainder theorem and Euclidean algorithm and their applications</td>
</tr>
<tr>
<td><strong>3.</strong> To resolve rational functions into partial fractions</td>
</tr>
<tr>
<td><strong>4.</strong> To learn the properties of roots of polynomial equations with real coefficients in one variable</td>
</tr>
<tr>
<td>A7</td>
</tr>
<tr>
<td>---</td>
</tr>
</tbody>
</table>
| A8 | Matrices  
8.1 Matrices and their operations  
8.2 Square matrices of order 2 and 3  
8.3 Applications to two dimensional geometry |
|   | 1. To learn the concept and operations of matrices  
2. To learn the properties and operations of square matrices of order 2 and 3 and determinants  
3. To apply matrices to two dimensional geometry |
| A9 | System of Linear Equations in 2 or 3 Unknowns  
9.1 Gaussian elimination and Echelon form  
9.2 Existence and uniqueness of solution |
|   | 1. To solve a system of linear equations using Gaussian elimination  
2. To recognize the existence and uniqueness of solution |
| A10 | Complex Numbers  
10.1 Definition of complex numbers and their arithmetic operations  
10.2 Argand diagram, argument and conjugate  
10.3 Simple applications in plane geometry  
10.4 De Moivre's theorem |
|   | 1. To learn the properties of complex numbers, their geometrical representations and applications  
2. To learn the De Moivre's Theorem and its applications in finding the nth roots of complex numbers, in solving polynomial equations and proving trigonometric identities |
| B1 | Sequence, Series and their Limits  
1.1 Sequence and series  
1.2 Limit of a sequence and series  
1.3 Convergence of a sequence and series |
|   | 1. To learn the concept of sequence and series  
2. To understand the intuitive concept of the limit of sequence and series  
3. To understand the behaviour of infinite sequence and series |
| B2 | Limit, Continuity and Differentiability  
2.1 Limit of a function  
2.2 Continuity of a function  
2.3 Differentiability of a function |
|   | 1. To understand the intuitive concept of the limit of a function  
2. To understand the intuitive concept of continuity and differentiability of a function  
3. To recognize limit as a fundamental concept in calculus |
| B3 | Differentiation  
3.1 Fundamental rules for differentiation |
<p>|   | 1. To acquire different techniques of differentiation |</p>
<table>
<thead>
<tr>
<th>Section</th>
<th>Subsection</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.2</td>
<td>Differentiation of trigonometric functions</td>
<td>2. To learn and acquire techniques to find higher order derivative</td>
</tr>
<tr>
<td>3.3</td>
<td>Differentiation of composite functions and inverse functions</td>
<td>3. To understand the intuitive concept of Rolle’s Theorem and Mean Value Theorem</td>
</tr>
<tr>
<td>3.4</td>
<td>Differentiation of implicit functions</td>
<td></td>
</tr>
<tr>
<td>3.5</td>
<td>Differentiation of parametric equations</td>
<td></td>
</tr>
<tr>
<td>3.6</td>
<td>Differentiation of logarithmic and exponential function</td>
<td></td>
</tr>
<tr>
<td>3.7</td>
<td>Higher order derivatives and Leibniz’s Theorem</td>
<td></td>
</tr>
<tr>
<td>3.8</td>
<td>The Rolle’s Theorem and Mean Value Theorem</td>
<td></td>
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</table>

<table>
<thead>
<tr>
<th>Section</th>
<th>Subsection</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>B4</td>
<td>Application of Differentiation</td>
<td>1. To learn and to use the L' Hospital’s Rule</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2. To learn the applications of differentiation</td>
</tr>
<tr>
<td>4.1</td>
<td>The L’Hospital’s Rule</td>
<td></td>
</tr>
<tr>
<td>4.2</td>
<td>Rate of change</td>
<td></td>
</tr>
<tr>
<td>4.3</td>
<td>Monotonic functions</td>
<td></td>
</tr>
<tr>
<td>4.4</td>
<td>Maxima and minima</td>
<td></td>
</tr>
<tr>
<td>4.5</td>
<td>Curve sketching</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Section</th>
<th>Subsection</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>B5</td>
<td>Integration</td>
<td>1. To understand the notion of integral as limit of a sum</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2. To learn some properties of integrals</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3. To understand the Fundamental Theorem of Integral Calculus</td>
</tr>
<tr>
<td></td>
<td></td>
<td>4. To apply the Fundamental Theorem of Integral Calculus in the evaluation of integrals</td>
</tr>
<tr>
<td></td>
<td></td>
<td>5. To learn the methods of integration</td>
</tr>
<tr>
<td>5.1</td>
<td>The Riemann definition of integration</td>
<td></td>
</tr>
<tr>
<td>5.2</td>
<td>Simple properties of definite integrals</td>
<td></td>
</tr>
<tr>
<td>5.3</td>
<td>The Mean Value Theorem for Integrals</td>
<td></td>
</tr>
<tr>
<td>5.4</td>
<td>Fundamental Theorem of Integral Calculus and its application to the evaluation of integrals</td>
<td></td>
</tr>
<tr>
<td>5.5</td>
<td>Indefinite integration</td>
<td></td>
</tr>
<tr>
<td>5.6</td>
<td>Method of integration</td>
<td></td>
</tr>
<tr>
<td>5.7</td>
<td>Improper integrals (deleted)</td>
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</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Section</th>
<th>Subsection</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>B6</td>
<td>Application of Integration</td>
<td>1. To learn the application of definite integration in the evaluation of plane area and volume of solid of revolution</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2. To apply definite integration to the evaluation of limit of sum</td>
</tr>
<tr>
<td>6.1</td>
<td>Plane area</td>
<td></td>
</tr>
<tr>
<td>6.2</td>
<td>Arc length (deleted)</td>
<td></td>
</tr>
<tr>
<td>6.3</td>
<td>Volume of revolution</td>
<td></td>
</tr>
<tr>
<td>6.4</td>
<td>Area of surface of revolution (deleted)</td>
<td></td>
</tr>
<tr>
<td>6.5</td>
<td>Limit of sum</td>
<td></td>
</tr>
</tbody>
</table>
B7 Analytical Geometry

7.1 Basic knowledge in coordinate geometry
7.2 Sketching of curves in the polar coordinate system (deleted)
7.3 Conic sections in rectangular coordinate system
7.4 Tangents and normals of conic sections
7.5 Locus problems in rectangular coordinate system
7.6 Tangents and normals of plane curves

1. To learn the conic sections
2. To study locus problems algebraically
3. To solve related problems

Suggested Sequence

There are two main topic areas in the curriculum and they are presented in the sequence as below.

<table>
<thead>
<tr>
<th>Topic Area A</th>
<th>Topic Area B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Algebra</td>
<td>Calculus and Analytical Geometry</td>
</tr>
<tr>
<td>Unit</td>
<td>Content</td>
</tr>
<tr>
<td>A1</td>
<td>The Language of Mathematics</td>
</tr>
<tr>
<td>A2</td>
<td>Functions</td>
</tr>
<tr>
<td>A3</td>
<td>Mathematical Induction</td>
</tr>
<tr>
<td>A4</td>
<td>Inequalities</td>
</tr>
<tr>
<td>A5</td>
<td>The Binomial Theorem for Positive Integral Indices</td>
</tr>
<tr>
<td>A6</td>
<td>Polynomials and Equations</td>
</tr>
<tr>
<td>A8</td>
<td>Matrices</td>
</tr>
<tr>
<td>A9</td>
<td>System of Linear Equations in 2 or 3 Unknowns</td>
</tr>
<tr>
<td>A10</td>
<td>Complex Numbers</td>
</tr>
</tbody>
</table>

(Note: The unit A7 has been deleted from the Syllabus 1992.)
Teachers should note that the sequence presented here only serves as an example and the categorization of the topics as A or B is done with a belief that such grouping and arrangement may offer a certain degree of fluency in teaching. In fact, teachers are free to design their own teaching sequence to suit the needs of their students. When designing a school-based curriculum of the subject, teachers should ensure that the curriculum should be coherent and students have already possessed the pre-requisite knowledge for the topics concerned. One possible sequence is as follows:

\[ A1 \rightarrow A2 \rightarrow A3 \rightarrow A4 \rightarrow A5 \rightarrow A6 \rightarrow B1 \]
\[ \rightarrow B2 \rightarrow B3 \rightarrow B4 \rightarrow B5 \rightarrow B6 \rightarrow B7 \rightarrow \]
\[ A8 \rightarrow A9 \rightarrow A10 \]

Some teachers, on the other hand, may prefer to apportion the number of periods allotted per week/cycle and start teaching according to the two sequences of topics in a “parallel” manner. Amongst different feasible approaches and sequencing of topics, teachers are expected to exercise their expertise in smoothing out, during teaching, possible irregularities sprung from the teaching sequence preferred. It is anticipated and advisable that the unit A1 “The Language of Mathematics” should be taught in the first place as a preliminary prerequisite so as to familiarize students with the usual symbols and trends of thinking in AL Pure Mathematics. The presentation in this curriculum and assessment guide will provide teachers with maximum flexibility so that the course of teaching adopted can be adjusted to meet the individual teaching situation.

To realize the spirit of the curriculum, teachers are advised to teach the curriculum as a connected body of mathematical knowledge as far as possible. Adequate arrangements should be provided for students to inquire, reason and communicate mathematically.

**Suggested Time Allocation**

The suggested time allocation for the course is 8 periods per week. It is assumed that there are 40 minutes in each period and 5 days in a week. A total of 380 periods (excluding the time spent on classroom tests and examinations) should be available for the two years. A time ratio is given to aid teachers in judging how far to take a given topic. This time ratio will indicate what fraction of the available total time may be spent on a certain unit, but schools are free to choose an equivalent or slightly different time allocation to suit their own situations. It can be seen, from the following table, that the total time ratio 312 is still 68
periods running short. This amount of time could be used for carrying out exploratory activities, consolidation activities or enrichment activities, etc. to suit the teaching approaches and the standard of students in the individual schools.

<table>
<thead>
<tr>
<th>Topic Area A</th>
<th>Topic Area B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Algebra</td>
<td>Calculus and Analytical Geometry</td>
</tr>
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<td>A3</td>
<td>Mathematical Induction</td>
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<td>A5</td>
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<td>A6</td>
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The following table shows the detailed breakdown of the units and the corresponding time ratios:

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<td>5.3 The Mean Value Theorem for Integrals</td>
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<td>5.4 Fundamental Theorem of Integral Calculus and its application to the evaluation of integrals</td>
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<td>5.5 Indefinite integration</td>
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<td>7.3 Conic sections in rectangular coordinate system</td>
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Sub-Total 160

Total (Topic Areas A and B) 312
Chapter 3  Learning and Teaching

Guiding Principles

In designing learning and teaching activities for AL Pure Mathematics, the following principles should be noted:

- Our main concern is to help students learn to learn rather than to deliver merely subject contents to students.
- All students can learn although at different paces.
- A learner-focused approach should be adopted.
- Information technology, when used appropriately, would increase the effectiveness of learning and teaching.

Suggested Learning and Teaching Strategies

Both the learning process and the end product in the learning of AL Pure Mathematics are important. Students should be allocated sufficient time to develop mathematical concepts, master problem solving skills and foster thinking abilities. It should be noted that students studying this curriculum are expected to have acquired mathematical knowledge at the Certificate of Education level (CE level), but previous knowledge of Additional Mathematics at the CE level is not necessary.

It should also be noted that no matter what emphasis of strategies is adopted, the teacher is the key person in classroom teaching. Liveliness and clear explanation of the teachers are students' main concern. Diversified learning and teaching activities are definitely beneficial to students. Applications of mathematical concepts to real-life situations also provide students motivation for learning AL Pure Mathematics.

In particular, attention should be paid to the following strategies when designing and preparing learning and teaching activities to facilitate students' learning.

Catering for Learner Differences

There is no hard and fast rule to address the problem of learner differences. However, the general approach of providing students with tasks or activities at different levels of difficulties seems viable. For less able students, tasks should be
relatively simple and fundamental in nature. For more able ones, tasks assigned should be challenging enough to cultivate and sustain their interest in learning. Alternatively, teachers could provide students with the same task, but vary the amount and style of supports, for example, giving more clues, breaking the more complicated problems into small parts for weaker students, etc.

The use of information technology (IT) could also provide another solution for teachers to cater for learner differences. For some topics, such as the limits of sequences and functions, sketching of curves, the use of appropriate software packages provides a simple, fast and accurate presentation which traditional teaching cannot fulfill. The use of IT in the learning of the subject is especially important to weaker students.

On the whole, no matter what strategy we are going to use to cater for learner differences, it should be able to maintain students' interest and confidence in learning mathematics.

**Appropriate Use of Information Technology**

For many years, lessons of AL Pure Mathematics have always been conducted with chalk and talk. Until recently, the wide spread use of computers provides enhancement for the learning and teaching in AL Pure Mathematics. Using IT in learning and teaching mathematics may bring about the following benefits:

(a) IT can enhance and extend mathematics learning experience, and encourage active student participation in exploratory and investigative activities.

(b) IT, when used as a tool, can support, supplement and extend learning and teaching activities, such as:
   - exercises and tutorials;
   - charting and graphical analysis;
   - simulation and modeling;
   - information retrieval and handling; and
   - data processing.

(c) IT may lead to new teaching strategies and practices in classrooms such as providing students with an interactive learning environment for contextual and situational learning.
IT in mathematics education could be considered as:

(i) a tool — Teachers could use presentation software to present notes, geometry software to demonstrate graphs and models, zoom-in and zoom-out facilities in some graphing calculators or graph plotter software to sketch the graphs of different functions. For example, teachers could use Excel programs to illustrate the limits of sequences and functions.

(ii) a tutor — Many mathematical software packages, in the form of CD-ROMs, could be served as a tutor to teach students mathematical concepts. These software packages illustrate mathematical concepts with texts, graphics and sound and contain graded exercises or tests. Students could use these software packages to revise the contents learnt in the classroom, remedy the weak areas or even learn new topics prior to teachers' teaching. They could further consolidate their learning with appropriate exercises chosen for their levels of difficulty at their own pace.

(iii) a tutee — Teachers could develop their own educational programs using spreadsheets or other programming languages to suit their own teaching strategies. Students could also make use of software to explore properties of curves.

Both teachers and students of AL Pure Mathematics are expected to decide when to use the available technology both intelligently and critically. For example, students have to decide whether to use graphing software or French curve to draw graphs of parabolas, ellipses and hyperbolas. Teachers have to decide whether to use computers or other devices for demonstration and which software is more appropriate for the task.

Besides, varieties of group work to facilitate collaborative learning or investigative approach in learning with IT should also be considered. Classwork or homework should emphasize concept development and understanding instead of manipulating complicated expressions or symbols or just rote memorization of formulae.
Appropriate Use of Multifarious Teaching Resources

Besides IT, there are other teaching resources that teachers could make use of in planning and conducting the learning and teaching activities:

- Reference books
- Learning and teaching packages
- Audio-visual tapes
- Instruments and other equipment for drawing shapes and making models
- Resources in libraries / resource centres, etc.

It is unlikely that a book / a series of books will cover all the topics of AI Pure Mathematics at an appropriate depth of treatment. Teachers should therefore exercise their discretion in selecting suitable parts from different books and reference materials to teach.

Mathematical language is progressively abstract. Different learning theories point out the importance of providing students with rich experiences in manipulating concrete objects as a foundation for the symbolic development. Teachers could make use of teaching aids such as simulation models, graph boards, etc., to demonstrate the mathematical concepts and allow students to “play” around before asking students to “structure and apply” the concepts.

A large quantity of related materials for teachers’ reference can be obtained from libraries or various resource centres operated by the Education and Manpower Bureau, such as Curriculum Resources Centres.

Internet is another popular source for sharing and retrieving information. Gathering and selecting information from these sources would be major learning activities in the 21st century.

Finally, teachers should note that this document is only a guide rather than a rigid teaching plan that must be followed closely. They are encouraged to explore and discover their own teaching methods and schedules appropriate to the ability level of their students.
Chapter 4    Assessment

Purposes of Assessment

It is generally agreed that assessment should promote students' learning and is an integral part of the learning–teaching cycle. The prime function of assessment has changed from providing a score or grade for ranking students to serving as an aid for learning. In general, assessment should be able to:

- provide reliable information that can be used to improve learning and teaching;
- provide feedback to students about their progress; and
- generate information to be used in reporting processes.

Clearly, assessment involves collecting, judging and interpreting information about students' performance. It can be formative or summative:

- Formative assessment is designed to measure what students know and are learning as they go along. The information collected is used as feedback to plan the future learning and teaching activities in which students and teachers are to be engaged. Formative assessment should be regular and ongoing and can be done in a number of ways including observations and discussions in class and examining students' written work done in class or at home.

- Summative assessment is designed to measure students' achievements and performance at certain intervals in time, such as at the end of a term or a school year. It is mainly used for providing a comprehensive and summary description of performance and progress in students' learning.

Both formative and summative assessments could achieve the said purposes. However, for diagnostic purpose aiming to identify students' strengths and weaknesses, it is imperative to make assessment on a regular basis. Therefore, formative assessment becomes more and more important in the learning and teaching process. There is nothing new as all teachers make regular assessments in the classes they teach. Most teachers would ask students questions, request them to attempt some questions either on the blackboard or at their seats,
hold discussions, organize class activities, etc. All these are formative assessments. What are highlighted here is that information should be collected regularly for making improvements to both learning and teaching and eventually raising standards. In general, assessment should not be considered as a separate add-on activity, but as an integral part of the learning and teaching cycle.

School Assessment

School assessment refers to all kinds of assessment activities that are administered in schools. It flexibly allows teachers to gather information to find out students' achievements related to the set objectives so as to make professional judgements about students' progress and to enhance the learning and teaching processes.

AL Pure Mathematics involves a wide range of learning objectives and processes. To gain a comprehensive understanding of student progress and achievement, evidence of student learning should be collected by a variety of assessment activities matching with the learning objectives. Both the processes (such as the strategies involved in solving a problem) and the products of learning (such as the solutions to problems) are important in mathematics learning. These should be reflected in the design of assessment. Different modes of assessment serve different purposes. Various assessment activities are needed to provide teachers with opportunities to collect, judge and interpret information about students' performance. Teachers should let students know how they will be assessed. Some common school assessment activities in AL Pure Mathematics include:

- **Class discussions or oral presentations**

  Class discussions and oral presentations are effective assessment activities. In the learning and teaching process, discussions, questioning and answering between the teacher and students (or among students) are often involved. Discussions in class not only enable teachers to discover what students understand about a particular topic, but also provide opportunities for students to present their views. They help foster their communication skills. Problems suitable for discussion include:

  ➢ Can you describe what happens to the curve \( y = \frac{x^3}{x^2 - 1} \) as \( x \) tends to infinity?

  ➢ It is true that all convergent sequences are bounded. Is the converse true? In other words, are all bounded sequences convergent?
Why \( \int_a^b f(x) \, dx \geq \int_a^b g(x) \, dx \) if \( f(x) \geq g(x) \) for all values in \([a, b]\)?

What is the meaning of partial fractions? Can you express \( \frac{x^4}{(x-k)(x^2+k^2)} \) into partial fractions?

What is the definition of an asymptote of a curve? How many types of asymptotes are associated with curve sketching? What are their characteristics?

Is the sequence \( a_n = (-1)^n (1 + \frac{1}{n}) \) oscillatory? Why?

Given that \( f(x) = x^3 \) and \( f'(0) = 0 \). Is it sufficient to conclude that \( f(x) \) has a relative extremum at \( x=0 \)? Why?

• Observations of students' performance in class

Observations of students' performance in class are useful assessment activities. It is not easy to judge progress and achievement in the development of thinking abilities (e.g. high order thinking skills) and attitudes. However, through observations (particularly long time observations), teachers can develop an ever-clearer picture of students' performance. Some criteria may be used for assessing students during observations. These include:

- Are students able to answer questions raised by teacher and peers?
- Can students present their solutions properly?
- Can students explain how they have arrived at the solutions and what strategies they have employed?
- What is the degree of students' participation in class?
- Do students raise sensible questions?
- Do students raise questions actively?
- What are students' attitudes, e.g. independence, cooperation and perseverance to work?

• Classwork and homework

Assignments such as classwork and homework are widely used in the learning and teaching processes and can help students consolidate concepts in mathematics and
teachers assess the performance of their students. It is important to give appropriate amount of assignments and to ensure that they are at a suitable level of difficulty. Each assignment should be appropriately related to specific objectives. It is inappropriate, for example, to give students an assignment which involves extremely difficult skills in evaluating an integral while the aim is to assess the application of definite integration in finding plane areas. Moreover, assignments should not be confined to routine mathematical problems. They should include reading mathematics reference books, preparatory work for discussions in class, searching the Internet and looking up newspapers, magazines and journals, etc. When marking classwork/homework, specific, clear, constructive and supportive comments, feedback and suggestions for improvement should be given. This kind of information tells students about their strengths, weaknesses, progress and enables them to know what they should do next in order to improve.

- **Project work**

Project learning is a powerful learning and teaching strategy to promote self-directed and self-regulated learning. It is not intended to replace the learning and teaching of subject knowledge in a discipline but provides an alternative learning experience, which allows students to have more space for learning. It enables students to construct and connect knowledge, skills, values and attitudes through a variety of activities. It is also a good vehicle for facilitating students' development of generic skills\(^3\). Therefore, project is a very useful activity to assess students' performance. Teachers should note that projects can be done individually or in groups depending on their nature. Students' performance in project work may be assessed using the following criteria:

1. Comprehension of the project
2. Use of strategy and approach
3. Coverage, depth, accuracy of content
4. Presentation and communication
5. Attitude

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\(^3\) The 9 essential generic skills identified are collaboration skills, communication skills, creativity, critical thinking skills, information technology skills, numeracy skills, problem solving skills, self-management skills and study skills.
• **Short quizzes**

Short quizzes can be conducted during a lesson as a revision. Students' responses often provide clues to their misunderstanding, levels of understanding, strengths, weaknesses, abilities, etc. Teachers can pose one or two simple problems on the topics previously taught (e.g. students are asked to evaluate a determinant with two rows identical or proportional) and assess from students' solutions their understanding on that topic (e.g. students are asked to evaluate a determinant by elementary row operations with all steps clearly shown). Short quizzes can sometimes be done in oral form.

• **Investigations**

Investigation is one type of class activity. When students conduct investigations, teachers can look at students' problem-solving skills and, if the activities are conducted in groups, collaboration skills. Students' performance during investigations can be assessed through observations. Criteria on assessing investigations include:

1. Comprehension of the problem
2. Use of strategies and approaches
3. Degree of participation and attitude

• **Tests and examinations**

Tests and examinations have been widely employed as the major methods of assessment within schools. Nevertheless, teachers should pay attention to the following points when setting test/examination papers:

1. The coverage in the paper should be proper and the item formats should be diversified.
2. Each item should have a clear assessment objective or objectives. Teachers should constantly refer to the curriculum aims and objectives when setting test and examination items.
3. Teachers should avoid testing only basic information recall and should try to construct items that assess the understanding of concepts, problem solving skills and high order thinking skills.
4. The item difficulty level should reflect students' abilities.
5. The language used in the paper should be simple and clear.
Before setting a test/examination paper, teachers should prepare a table of specifications and a marking scheme. In the table of specifications, marks allocation on the learning units being assessed should be clearly shown. Appropriate amount of marks should be allocated so as to reflect the aims and focus of the paper and to ensure the proper coverage of the topics being assessed. The paper should embrace various types of items, like short items, long items, structured items, etc. to assess students' knowledge in various aspects of mathematics. Open-ended questions should also be included to assess students' thinking abilities including communicating and reasoning skills in AL Pure Mathematics. Open-ended questions focus on students' understanding and their ability to reason and apply knowledge in less routine contexts. Such questions can reflect more clearly the levels of student achievement. In general, open-ended questions require complex thinking and may yield multiple solutions. They require teachers to interpret and use multiple criteria in evaluating students' responses. Instead of simple memorization, they require students to construct their own responses (e.g. construct a sequence which converges to zero; construct a non-zero $3 \times 3$ matrix so that its inverse does not exist) and hence open a window to students' thinking and understanding. Such tasks become vehicles for communicating students' actual achievements to parents, teachers and students themselves.

In summary, a balanced assessment program including a variety of valid assessment activities is necessary for assessing achievement of the general objectives.

**Feedback from Assessment**

Feedback is a crucial element of assessment. Effective feedback should help students recognise their next step in the process of learning, how to carry it forward and provide encouragement. It should also help teachers recognise the gaps between students' actual and expected performances, identify students' strengths and areas for improvement, and improve teaching practice.

Teachers can use the information collected in the formative assessment activities to adjust teaching strategies, decide whether to include further consolidation activities or introduce enrichment topics in subsequent day-to-day teaching.

Feedback from summative assessment activities can provide information for students to plan their subsequent phase of study and teachers to plan the teaching sequence, and to adjust the
breath and depth of the curriculum for the subsequent term or year. This information can be very useful for schools to adjust the aims and strategies of the school-based curriculum of AL Pure Mathematics.

Public Assessment

Hong Kong has relied on written tests and examinations as major methods of public assessment as well as within schools. Written tests and examinations assess the products of learning such as memory, understanding of knowledge and concepts at a certain point in time. The Hong Kong Examinations and Assessment Authority (HKEAA) organizes the public assessment on AL Pure Mathematics curriculum to assess students' attainment on the aims and objectives. The public assessment serves to provide a testing of all students for the purposes of certification and selection. Moreover, the public assessment can also generate useful feedback on the effectiveness of learning and teaching of the subject through the subject report which provides students' overall performance in the examination.

The AL Pure Mathematics is the subject that has been designed for students intending to continue their studies in mathematics, engineering, science and technology. Students studying this subject are expected to have acquired mathematical knowledge at the Certificate of Education (CE) level in the subject Mathematics.

The assessment objective of the public examination is to test the understanding of basic mathematical concepts and their applications. The formats and details of the public examination can be found in the Handbook “Hong Kong Advanced Level Examination Regulations and Syllabuses” published annually by the Hong Kong Examinations and Assessment Authority.
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Chapter 5  Exemplars

Exemplar 1  A. M. $\geq$ G. M.

Exemplar 2  Plane Area

Exemplar 3  The Binomial Theorem
Exemplar 1:

A. M. ≥ G. M.

Objective: To prove A. M. ≥ G. M. without the application of Backward Induction

Pre-requisite knowledge: (1) The Principle of Mathematical Induction
(2) Fundamental techniques in proving absolute inequalities

Description of the Activity:

Let $A_n = \frac{a_1 + a_2 + \cdots + a_n}{n}$ and $G_n = (a_1 a_2 \cdots a_n)^{1/n}$, where $a_1, a_2, \ldots, a_n$ are $n$ positive numbers. It was suggested in the Syllabuses for Secondary Schools – Pure Mathematics (Advanced Level) 1992 that teachers may prove $A_n > G_n$ by backward induction if required. However, backward induction is deleted from this Curriculum. Some suggestions to prove the inequality are as follows:

Method 1

It is obvious that $A_1 = G_1$ and $A_2 > G_2$.

Assume that $A_k > G_k$ is true, where $k$ is a positive integer greater than or equal to 2.

When $n = k + 1$,

Case (i) If $a_1 = a_2 = \cdots = a_{k+1}$, then $A_{k+1} = G_{k+1}$.

Case (ii) If not all $a_1, a_2, \ldots, a_{k+1}$ are equal, we may assume, without loss of generality, that $a_1 \leq a_2 \leq \cdots \leq a_{k+1}$ and $a_1 < a_{k+1}$.

It follows that $\frac{a_j}{G_{k+1}} < 1$, $\frac{a_{k+1}}{G_{k+1}} > 1$.

Let $y = a_1 a_{k+1}$. Since $A_k > G_k$, we have

$$\frac{y}{(G_{k+1})^2} + \frac{a_2}{G_{k+1}} + \cdots + \frac{a_k}{G_{k+1}} \geq k \sqrt[n]{\frac{y}{(G_{k+1})^2} \cdot \frac{a_2}{G_{k+1}} \cdots \frac{a_k}{G_{k+1}} G_{k+1}^{-1}} = k \sqrt[n]{\frac{a_1 a_2 \cdots a_k}{(G_{k+1})^{-1}}} = k$$

Adding $\frac{a_1}{G_{k+1}} + a_{k+1} - \frac{y}{(G_{k+1})^2}$ to both sides of the inequality, we have

$$\frac{a_1}{G_{k+1}} + \frac{a_2}{G_{k+1}} + \cdots + \frac{a_k}{G_{k+1}} + a_{k+1} \geq k + \frac{a_1}{G_{k+1}} + a_{k+1} - \frac{y}{(G_{k+1})^2}$$

$$= k + 1 + \frac{a_1}{G_{k+1}} - 1 + \frac{a_{k+1}}{G_{k+1}} - \frac{a_1 a_{k+1}}{G_{k+1} G_{k+1}} - \frac{1}{G_{k+1}}$$
Thus, we have 
\[H_{k+1} = k + 1 + \left(1 - \frac{a_1}{G_{k+1}}\right) \left(\frac{a_{k+1}}{G_{k+1}} - 1\right)\]

\[\therefore \frac{a_1}{G_{k+1}} < 1, \quad \frac{a_{k+1}}{G_{k+1}} > 1, \quad \therefore \left(1 - \frac{a_1}{G_{k+1}}\right) \left(\frac{a_{k+1}}{G_{k+1}} - 1\right) > 0.\]

Thus, we have 
\[\frac{a_1}{G_{k+1}} + \frac{a_2}{G_{k+1}} + \ldots + \frac{a_k}{G_{k+1}} + \frac{a_{k+1}}{G_{k+1}} > k + 1.\]

\[\therefore A_{k+1} > G_{k+1} \text{ holds.}\]

From cases (i) and (ii), we have \(A_{k+1} \geq G_{k+1}\).

By the principle of mathematical induction, \(A_n \geq G_n\) is true for all natural numbers \(n\).

**Method 2**

It is obvious that \(A_1 = G_1\) and \(A_2 \geq G_2\).

Assume that \(A_k \geq G_k\) is true, where \(k\) is a positive integer greater than or equal to 2.

Let the geometric mean and the arithmetic mean of \(A_{k+1}, A_{k+1}, \ldots, A_{k+1}\) be \(M\) and \(L\) respectively. Then \(M = (a_{k+1}A_{k+1}^{-1})^\frac{1}{k-1}\) and \(L = \frac{1}{k} [a_{k+1} + (k-1)A_{k+1}]\).

By the induction hypothesis, \(M \leq L\) and

\[(G_{k+1}^{-1}A_{k+1}^{-1})^\frac{1}{k-1} = (a_1a_2\ldots a_k)^{\frac{1}{k}} (a_{k+1}A_{k+1}^{-1})^\frac{1}{k-1} = \left[\frac{1}{k}M\right]^\frac{1}{k-1}\]

\[\leq \frac{1}{2} (G_{k+1} + M)\]

\[\leq \frac{1}{2} (A_{k+1} + L) \quad \text{since} \quad G_k \leq A_k \quad \text{and} \quad M \leq L\]

\[= \frac{1}{2} \left\{A_k + \frac{1}{k} [a_{k+1} + (k-1)A_{k+1}]\right\}\]

\[= \frac{1}{2k} \left\{a_{k+1} + ka_k + (k-1)A_{k+1}\right\}\]

\[= \frac{1}{2k} \left\{(k+1)A_{k+1} + (k-1)A_{k+1}\right\}\]

\[= A_{k+1}\]

i.e. \((G_{k+1})^{k+1}(A_{k+1})^{-k-1} \leq (A_{k+1})^{2k}\)

\[\therefore G_{k+1} \leq A_{k+1}\] holds.

By the principle of mathematical induction, \(A_n \geq G_n\) is true for all natural numbers \(n\).
Method 3

It is obvious that \( A_1 = G_1 \) and \( A_2 \geq G_2 \).

Assume that \( A_n \geq G_n \) is true, where \( k \) is a positive integer greater than or equal to 2.

When \( n = k + 1 \),

Case (i) If \( a_1 = a_2 = \cdots = a_{k+1} \), then \( A_{k+1} = G_{k+1} \).

Case (ii) If not all \( a_1, a_2, \ldots, a_{k+1} \) are equal, we may assume, without loss of generality, that \( a_1 \leq a_2 \leq \cdots \leq a_{k+1} \) and \( a_1 < a_{k+1} \).

Since \( A_n \geq G_n \), we have

\[
a_1 + a_2 + \cdots + a_k + a_{k+1} \geq k \sqrt[k]{a_1 a_2 \cdots a_k a_{k+1}}
\]

\[
= k \sqrt[k]{a_1 a_2 \cdots a_{k+1}} + 1
\]

\[
= (k^{k+1} + 1) a_{k+1}
\]

where \( r^{k+1} = \frac{a_1 a_2 \cdots a_{k+1}}{(a_{k+1})^{k+1}} \) and \( r^{k+1} < 1 \) with \( r > 0 \).

Since \( 0 < r^{k+1} < 1 \), then \( 0 < r < 1 \) and \( r^k < r^{k-1} < r^{k-2} < \ldots < r^3 < r^2 < r \).

As

\[
1 - \frac{r^{k+1}}{1 - r} = 1 + r + \cdots + r^k \geq r^k + r^{k-1} + \cdots + r^2 = (k+1)r^k,
\]

\[
\therefore \ 1 - r^{k+1} > (k+1)r^k (1 - r).
\]

Hence, we have

\[
1 - r^{k+1} + (k+1)r^k > (k+1)\]

\[
\therefore \ k^{k+1} + 1 > (k+1)\]

Since \( a_1 + a_2 + \cdots + a_k + a_{k+1} \geq (k^{k+1} + 1) a_{k+1} \)

\[
\therefore \ a_1 + a_2 + \cdots + a_k + a_{k+1} \geq (k+1)r^k a_{k+1}
\]

\[
= (k+1) \sqrt[k]{a_1 a_2 \cdots a_{k+1}} a_{k+1}
\]

\[
= (k+1) \sqrt[k]{a_1 a_2 \cdots a_{k+1}} a_{k+1}
\]

\[
\therefore \ A_{k+1} \geq G_{k+1} \] holds.

From cases (i) and (ii), we have \( A_{k+1} \geq G_{k+1} \).

By the principle of mathematical induction, \( A_n \geq G_n \) is true for all natural numbers \( n \).
Method 4

It is obvious that $A_1 = G_1$ and $A_2 \geq G_2$.

Assume that $A_k \geq G_k$ is true, where $k$ is a positive integer greater than or equal to 2.

When $n = k + 1$,

Case (i) If $a_1 = a_2 = \cdots = a_{k+1}$, then $A_{k+1} = G_{k+1}$.

Case (ii) If not all $a_1, a_2, \ldots, a_{k+1}$ are equal, we may assume, without loss of generality, that $a_1 \leq a_2 \leq \cdots \leq a_{k+1}$ and $a_i < a_{k+1}$.

It follows that $a_{k+1} > \sqrt[k+1]{a_1 a_2 \cdots a_k} = G_k$, and so $A_{k+1} - G_k > 0$.

By the induction hypothesis,

$$
A_{k+1} = \frac{1}{k+1} \left( a_1 + a_2 + \cdots + a_k + a_{k+1} \right) \\
= \frac{ka_k + a_{k+1}}{k+1} \\
\geq \frac{kG_k + a_{k+1}}{k+1} \\
= G_k + \frac{a_{k+1} - G_k}{k+1}
$$

By the Binomial Theorem, we have

$$
(A_{k+1})^{k+1} \geq \left( G_k + \frac{a_{k+1} - G_k}{k+1} \right)^{k+1} \\
= (G_k)^{k+1} + (k+1)(G_k)^k \left( \frac{a_{k+1} - G_k}{k+1} \right) + \cdots \quad \text{(all terms are positive)} \\
> (G_k)^{k+1} + (G_k)^k (a_{k+1} - G_k) \\
= (G_k)^k a_{k+1} \\
= (G_{k+1})^{k+1}
$$

$\therefore A_{k+1} > G_{k+1}$ holds.

From cases (i) and (ii), we have $A_{k+1} \geq G_{k+1}$.

By the principle of mathematical induction, $A_n \geq G_n$ is true for all natural numbers $n$. 

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Exemplar 2:

**Plane Area**

**Objective:** To prove that the area bounded by the curve with parametric equations \( x = x(t) \), \( y = y(t) \), and the lines OA, OB is

\[
\frac{1}{2} \int_{t_0}^{t_1} (x \frac{dy}{dt} - y \frac{dx}{dt}) dt
\]

\[\text{..................(*)}\]

where the parameters of A and B are \( t_0 \) and \( t_1 \) respectively.

**Pre-requisite knowledge:**

1. The application of definite integrals to find the area under a curve in Cartesian form.
2. The Fundamental Theorem of Integral Calculus.

**Description of the Activity:**

Many teachers used to prove the formula (*) by means of formulae related to the polar coordinate system. The formula (*) can be readily derived from \( \frac{1}{2} \int_\alpha^\beta r^2 \ d\theta \), which gives the area bounded by the curve with the polar equation \( r = f(\theta) \) and the two radii with radius vectors \( \theta = \alpha \) and \( \theta = \beta \). The contents related to the polar coordinate system are deleted from this curriculum. A suggestion to prove the formula (*) is as follows:

A curve with the parametric equations \( x = x(t) \), \( y = y(t) \) is shown in the diagram above. The parameters of A and B are \( t_0 \) and \( t_1 \) respectively. Without loss of generality, we may assume that, when the parameter \( t \) increases, the curve is continuous and goes in the anticlockwise direction.
Area of the shaded region = Area of Δ BOC + Area of ABCD - Area of Δ AOD

\[ = \frac{x(t_1) - x(t_2)}{2} + \int_{t_1}^{t_2} y(t) \, dx - \frac{x(t_2) - x(t_1)}{2} + \int_{t_1}^{t_2} y'(t) \, dx - \frac{x(t_2) - x(t_1)}{2} \]

\[ = \frac{1}{2} \left[ x(t_1) y(t_1) - x(t_2) y(t_2) \right] - \int_{t_1}^{t_2} y(t) x'(t) \, dt \]

By the Second fundamental Theorem of Integral Calculus (p. 69 in Appendix 2), we have

\[ x(t_1) y(t_1) - x(t_2) y(t_2) = \int_{t_1}^{t_2} d[x(t) y(t)] \]

Hence, we have

Area of the shaded region \(= \frac{1}{2} \int_{t_1}^{t_2} d[x(t) y(t)] - \int_{t_1}^{t_2} y(t) x'(t) \, dt \)

\[ = \frac{1}{2} \left[ \int_{t_1}^{t_2} [x(t) y'(t) + x'(t) y(t)] \, dt - \int_{t_1}^{t_2} y(t) x'(t) \, dt \right] \]

\[ = \frac{1}{2} \left[ \int_{t_1}^{t_2} [x(t) y'(t) - x'(t) y(t)] \, dt \right] \]

\[ = \frac{1}{2} \left[ \int_{t_1}^{t_2} \left[ x(t) \frac{dy}{dt} - y(t) \frac{dx}{dt} \right] \, dt \right] \]

i.e. The area bounded by the curve with parametric equations \(x=x(t), y=y(t)\) and the lines OA,

OB is \(\frac{1}{2} \int_{t_1}^{t_2} \left[ x(t) \frac{dy}{dt} - y(t) \frac{dx}{dt} \right] \, dt \).
Exemplar 3:

The Binomial Theorem

Objective: To prove the Binomial Theorem for positive integral indices.

Pre-requisite knowledge: The relations between the roots and coefficients of a polynomial equation with real coefficients.

Description of the Activity:

Most teachers apply the Principle of Mathematical Induction to prove the Binomial Theorem for positive integral indices. An alternative way to prove that, for positive integers n,

\[(a+b)^n = C_0^n a^n + C_1^n a^{n-1} b + C_2^n a^{n-2} b^2 + \ldots + C_{n-1}^n a b^{n-1} + C_n^n b^n\]

is as follows:

Let \((x+b)^n = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0\), where \(a_0, a_1, \ldots, a_{n-1}, a_n\) are real constants.

Since the equation \((x+b)^n = 0\) has \(n\) repeated roots \(x = -b\), the equation \(a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = 0\) has \(n\) roots \(x_1, x_2, \ldots, x_{n-1}, x_n\) with \(x_1 = x_2 = \ldots = x_{n-1} = x_n = -b\).

By using the relations between the roots and coefficients of a polynomial equation with real coefficients,

\[
\begin{align*}
&x_1 + x_2 + \ldots + x_n = -\frac{a_{n-1}}{a_n}, \\
&x_1 x_2 + x_1 x_3 + \ldots + x_{n-1} x_n = \frac{a_{n-2}}{a_n}, \\
&x_1 x_2 x_3 + x_1 x_2 x_4 + \ldots + x_{n-2} x_{n-1} x_n = -\frac{a_{n-3}}{a_n}, \\
&\ldots \ldots \\
&x_1 x_2 \ldots x_k + x_1 x_2 \ldots x_{k+1} + \ldots + x_{n-k+1} x_{n-k+2} \ldots x_n = (-1)^k \frac{a_{n-k}}{a_n}, \\
&\ldots \ldots \\
&x_1 x_2 \ldots x_n = (-1)^n \frac{a_0}{a_n}.
\end{align*}
\]
In the $k^{th}$ equality above, the left-hand side is the sum of the product of $k$ terms of $x_i$.

Since $x_1 = x_2 = ... = x_{n-1} = x_n = -b$

\[ \therefore (-b)^k C_k^n = (-1)^k \frac{a_n b^n}{a_n} \]

As $a_n$ is the coefficient of $x_n$ in the expansion of $(x+b)^n$, it is obvious that $a_n = 1$,

\[ \therefore a_{n-k} = C^n_k b^k \quad (k = 1, 2, \ldots, n). \]

i.e. $a_n = 1 = C^n_0$, $a_{n-1} = C^n_1 b$, $\ldots$, $a_{n-k} = C^n_k b^k$, $\ldots$, $a_0 = C^n_n b^n$

\[ \therefore (x+b)^n = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ldots + a_k x^k + \ldots + a_0 x + a_0 \]

\[ = C^n_0 x^n + C^n_1 b x^{n-1} + C^n_2 b^2 x^{n-2} + \ldots + C^n_{n-k} b^{n-k} x^k + \ldots + C^n_{n-1} b^{n-1} x + C^n_n b^n. \]

Putting $x = a$, we have

\[ (a+b)^n = C^n_0 a^n + C^n_1 b a^{n-1} + C^n_2 b^2 a^{n-2} + \ldots + C^n_{n-k} b^{n-k} a^k + \ldots + C^n_{n-1} b^{n-1} a + C^n_n b^n \]

\[ = C^n_0 a^n + C^n_1 a^{n-1} b + C^n_2 a^{n-2} b^2 + \ldots + C^n_{n-k} a^k b^{n-k} + \ldots + C^n_{n-1} a b^{n-1} + C^n_n b^n. \]

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## Appendix 1

**Summary of Changes to the Contents of Syllabuses for Secondary Schools – Pure Mathematics (Advanced Level) 1992**

<table>
<thead>
<tr>
<th>Unit</th>
<th>Sub-unit</th>
<th>Topics</th>
<th>Content affected</th>
<th>Time Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>A3</td>
<td>3.2</td>
<td>Other common variations of the Principle of Mathematical Induction and their applications</td>
<td>“Backward induction” is deleted</td>
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<tr>
<td>A4</td>
<td>4.2</td>
<td>A.M. ≥ G.M.</td>
<td>The sentence “If required, teachers may apply backward induction” is deleted</td>
<td>0</td>
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<tr>
<td>A5</td>
<td>5.2</td>
<td>Applications of the binomial theorem for positive integral indices</td>
<td>The sentence “Students are expected …… should be discussed” is deleted</td>
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<tr>
<td>A7</td>
<td>7.1 – 7.6</td>
<td>Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$</td>
<td>The whole unit is deleted</td>
<td>31</td>
</tr>
<tr>
<td>A9</td>
<td>9.2</td>
<td>Existence and uniqueness of solution</td>
<td>The sentence “The corresponding geometrical meaning …..coordinate geometry” is deleted</td>
<td>0</td>
</tr>
<tr>
<td>A10</td>
<td>10.4c</td>
<td>$n^{th}$ roots of a complex number and their geometrical interpretation</td>
<td>The sentence “4. Factorize $z^{2n} - 2z^n \cos n\theta + 1$ into real quadratic factors” is deleted</td>
<td>1</td>
</tr>
<tr>
<td>B5</td>
<td>5.7</td>
<td>Improper integrals</td>
<td>The sub-unit is deleted</td>
<td>4</td>
</tr>
<tr>
<td>B6</td>
<td>6.2</td>
<td>Arc length</td>
<td>The sub-unit is deleted</td>
<td>3</td>
</tr>
<tr>
<td>B6</td>
<td>6.4</td>
<td>Area of surface of revolution</td>
<td>The sub-unit is deleted</td>
<td>4</td>
</tr>
<tr>
<td>B7</td>
<td>7.1</td>
<td>Basic knowledge in coordinate geometry</td>
<td>The paragraph “Students should be able to make conversions between polar and rectangular coordinate systems…….in which (x,y) lies” is deleted</td>
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<tr>
<td></td>
<td>7.2</td>
<td>Sketching of curves in the polar coordinate system</td>
<td>The sub-unit is deleted</td>
<td>4</td>
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Comparison between

**Pure Mathematics Curriculum and Assessment Guide**

*(Advanced Level)*

and

**Syllabuses for Secondary Schools – Pure Mathematics**

*(Advanced Level) 1992*

This Pure Mathematics Curriculum and Assessment Guide (Advanced Level) is a revised edition of the version published in 1992. Some topics have been deleted or trimmed. For ease of reference of teachers, these topics are enclosed in boxes like [ ] from the 1992 version, page for page. Notes and additional remarks are enclosed in boxes like [ ] to delimit the complexity of teaching.
UNIT A1: The Language of Mathematics

Objective: (1) To understand the first notion of set language.
(2) To understand the first notion of logic.

<table>
<thead>
<tr>
<th>Detailed Content</th>
<th>Time Ratio</th>
<th>Notes on Teaching</th>
</tr>
</thead>
</table>
| 1.1 Set language    | 5          | Basic terminology to be introduced includes set, element, subset, mother set, power set, empty (void) set, equal sets, disjoint sets, universal set, intersection, union, complement, and product of sets. It should be noted that in ushering in the foregoing concepts, just informal treatment is expected and teachers are encouraged to adopt an adequately wide spectrum of simple and factual examples of daily life nature to support their teaching. Conventionally used symbols and notations should be clearly taught. The following are for reference.
(1) Sets are generally denoted by capital letters and elements by small letters. The sets of numbers listed below are commonly denoted by the accompanying symbols:
- the set of all natural numbers \( N \)
- the set of all integers \( Z \)
- the set of all rational numbers \( Q \)
- the set of all real numbers \( R \)
- the set of all complex numbers \( C \)
(2) Sets are usually presented either in tabular form i.e. with all the elements listed out like \( A = \{ 2, 4, 6, 8, 10 \} \) or in propositional form \( \{ x : p(x) \} \) like \( A = \{ x : x \leq 10, x \text{ is an even and positive integer} \} \)
(3) Just simple and straightforward operation rules on intersection, union and complement may be introduced to substantiate students' learning. It is advisable to use Venn diagrams to offer intuitive understanding of the rules as it is probably the first time for the students to come across terms like commutative, associative and distributive etc. |

<table>
<thead>
<tr>
<th>Detailed Content</th>
<th>Time Ratio</th>
<th>Notes on Teaching</th>
</tr>
</thead>
</table>
| 1.2 Simple logic    | 5          | Basic terminology to be introduced includes statement/proposition, truth value, conjunction, disjunction, negation, conditional and biconditional, equivalent statements, equivalence, implication, quantifiers, examples and counter-examples. The use of truth table to manifest the meaning of the above connectives is an advisable approach. It is anticipated that more emphasis will be directed to the teaching of conditional and biconditional in the form of "if-then" and "if-and-only-if" which are very widely used in the study of mathematics. Teachers should also touch upon "theorem" and "converse". In teaching this topic, teachers should provide adequate relevant daily life statements for illustration in the first place and then students should be encouraged to give examples of their own. It must be noted that class discussion with the students is helpful in bringing around the concept and using it as a tool. The pure analytical approach is not desirable.
To reinforce students' understanding regarding 'sufficient condition', 'necessary condition' and 'necessary and sufficient condition', teachers may 'lead a discussion with them using simple examples as follows which consist of two propositions and teachers may proceed to investigate with students which condition(s) is (are) applicable:
(1) \( x \) and \( y \) are integers; \( xy \) is an integer.
(2) \( x \) and \( y \) are even; \( xy \) is even.
(3) \( x \) and \( y \) are even; \( x+y \) and \( xy \) are even.
(4) The equation \( ax^2 + bx + c = 0 \) has equal roots; \( b^2 - 4ac = 0 \).
Furthermore, the fact that \( p \rightarrow q \equiv \sim q \rightarrow \sim p \) should be elaborated (commonly known as contrapositive) with reference to some simple results like the above-mentioned examples. Also teachers may demonstrate some proofs using the method of contradiction. For example, to prove \( \sqrt{2} \) is irrational is a commonplace. |

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## Unit A2: Functions

**Objective:**
1. To recognize function as a fundamental tool in other branches of mathematics
2. To sketch and to describe the shapes of different functions

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<tr>
<td><strong>2.1 Functions and their graphs</strong></td>
<td>2</td>
<td>Students should be taught with a clear definition of function, however rigorous treatment is not expected. The following version may be adopted: f: A → B, f is a function from A to B if every element in A associated with an unique element in B. A is called the domain of f, B the range of f. For an element x in A, the element in B which is associated with x under f usually denoted by f(x) is called the image of x under f and f[A] denotes the set of images of A under f. Special emphasis should be put on real-valued functions since they are most useful in the discussion of other mathematical topics in this syllabus. Students are also expected to be able to plot/sketch the graph of functions.</td>
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</table>
| **2.2 Properties and operations of functions** | 4 | Students should be taught with a clear definition of injective, surjective and bijective functions so that students are able to distinguish them and to apply the knowledge to solve problems concerned. The following suggested versions may be adopted: 
- A function f: A → B is
  1. injective (one-to-one) if and only if for elements a₁, a₂ in A, a₁ ≠ a₂ implies f(a₁) ≠ f(a₂), or equivalently, f(a₁) = f(a₂) implies a₁ = a₂.
  2. surjective (onto) if and only if f[A] = B, i.e., every element in B is the image of an element in A.
  3. bijective (one-to-one correspondence) if and only if f is injective and surjective. At this juncture, teachers may provide sufficient preparation on the part of the students so that the concept of inverse function denoted by f⁻¹ can be easily figured out and the property that f is injective if and only if its inverse function f⁻¹ exists.
Moreover the property that the graphs of a function and its inverse (if exists) are reflections about the line y = x should be studied with adequate illustrations. |
| | | These properties may be useful in sketching curves and in evaluating definite integrals, etc. Concerning the operations with functions, teachers should discuss with the students that, for functions f and g, f + g, f - g, f × g and f ÷ g (provided g(x) ≠ 0 for all values of x concerned) are again functions. However regarding the composition of functions which is very useful especially in teaching the chain rule in differential calculus, it is desirable to give ample ample examples so as to support students' mastery of the concepts. Teachers may consider the following suggestion with due emphasis directed to real-valued functions. If f: A → B and g: B → C are functions, then the composition of f and g is the function g ◦ f: A → C such that g ◦ f(x) = g(f(x)) for all element x in A. |

- **Diagram:**

  ![Diagram](image-url)
### Detailed Content | Time Ratio | Notes on Teaching
--- | --- | ---
2.3 Algebraic functions | 2 | Students should be able to recognize the following algebraic functions:
(a) polynomial functions
(b) rational functions
(c) power functions $x^a$ where $a$ is rational
(d) other algebraic functions derived from the above-mentioned ones through addition, subtraction, multiplication, division and composition like $\sqrt{x^2 + 1}$

Students should be able to recognize the following algebraic functions:
(a) polynomial functions
(b) rational functions
(c) power functions $x^a$ where $a$ is rational
(d) other algebraic functions derived from the above-mentioned ones through addition, subtraction, multiplication, division and composition like $\sqrt{x^2 + 1}$

2.4 Trigonometric functions and their formulae | 14 | Students should be able to sketch the graphs of the six trigonometric functions and their inverse functions. The basic relations like

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$\tan^2 \theta + 1 = \sec^2 \theta$$

$$\cot^2 \theta + 1 = \csc^2 \theta$$

should also be included in the discussion with the students. Simplification of these functions at $\frac{n\pi}{2} \pm \theta$ for odd and even $n$ and proving identities are expected. The knowledge and related applications of the following are also expected:

1. **Compound angle formulae**
   - \(\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B\)
   - \(\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B\)
   - \(\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}\)

2. **Multiple angle formulae**
   - \(\sin 2A = 2 \sin A \cos A\)
   - \(\cos 2A = \cos^2 A - \sin^2 A\)
   - \(\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}\)

3. **Half-angle formulae**
   - \(\sin^2 \frac{A}{2} = \frac{1}{2}(1 - \cos A)\)
   - \(\cos^2 \frac{A}{2} = \frac{1}{2}(1 + \cos A)\)
   - \(\sin A = \frac{2t}{1 + t^2}, \quad \cos A = \frac{1 - t^2}{1 + t^2}\)
   - with \(t = \tan \frac{A}{2}\)

4. **Sum and product formulae**
   - \(\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}\)
   - \(\sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}\)
   - \(\cos A + \cos B = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2}\)
   - \(\cos A - \cos B = 2 \sin \frac{A+B}{2} \sin \frac{B-A}{2}\)
   - \(\sin A \cos B = \frac{1}{2} (\sin (A+B) + \sin (A-B))\)
   - \(\cos A \sin B = \frac{1}{2} (\sin (A+B) - \sin (A-B))\)
   - \(\cos A \cos B = \frac{1}{2} (\cos (A+B) + \cos (A-B))\)
   - \(\sin A \sin B = \frac{1}{2} (\cos (A-B) - \cos (A+B))\)

As a matter of fact, most of the formulae are direct consequence from the basic ones thus teachers should encourage their students to work out the proofs for
2.5 Exponential and logarithmic functions

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<td>2.5 Exponential and logarithmic functions</td>
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<tr>
<td>2.5 Exponential and logarithmic functions</td>
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<td>themselves. The transformation of the expression $a \cos x + b \sin x$ into the form $r \sin(x + \alpha)$ or $r \cos(x + \beta)$ should be discussed. Applications of the above formulae in proving identities and in solving trigonometric equation should be included.</td>
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<td>Students should know the relation between the exponential and logarithmic functions, viz. one is the inverse function of the other. The common definition of logarithm, like</td>
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<td>$\log_a x = y$ if and only if $x = a^y$</td>
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<td>with $a &gt; 0$ and $a \neq 1$ should be revised. As for logarithmic functions of a variable $x$, students should know that they are functions of $\log x$ or of the logarithm of some function of $x$, like</td>
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<td>$(\log_b x)^2$ and $\log_a (1 + \tan x)$ etc.</td>
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<td>Some common properties of the logarithmic function should be studied:</td>
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<tr>
<td>For $f(x) = \log_a x$ with $a &gt; 0$ and $a \neq 1$:</td>
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<tr>
<td>(i) $f(x)$ is defined for $x &gt; 0$ only;</td>
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<tr>
<td>(ii) $f(x)$ is an increasing function if $a &gt; 1$ and is a decreasing function if $0 &lt; a &lt; 1$;</td>
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<td>(iii) for $b, c &gt; 0$ and $b \neq 1$, $\log_a c = \frac{\log_b c}{\log_b a}$;</td>
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<td>(iv) $f(a) = \log_a a = 1$;</td>
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<td>(v) $f(1) = \log_a 1 = 0$.</td>
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<td>For the exponential function, a parallel treatment should be provided, viz. a function of the form $a^x$ where $a$ is a positive constant and $x$ a variable is called an exponential function. Three common properties of the exponential function include the following:</td>
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<tr>
<td>For $f(x) = a^x$ with $a &gt; 0$ and $a \neq 1$:</td>
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<tr>
<td>(i) $f(x)$ is defined for all real $x$;</td>
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<tr>
<td>(ii) $f(x)$ is an increasing function if $a &gt; 1$ and is a decreasing one if $0 &lt; a &lt; 1$;</td>
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<td>(iii) $f(0)$ $= a^0 = 1$.</td>
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<tr>
<td>Students should be able to sketch the graph for</td>
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<tr>
<td>(i) the logarithmic function</td>
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<td>$f(x) = \log_a x$ for the cases $a &gt; 1$ and $0 &lt; a &lt; 1$</td>
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<tr>
<td>(ii) the exponential function also for the cases $a &gt; 1$ and $0 &lt; a &lt; 1$.</td>
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<td>At this juncture, teachers may embark on the following important results so as to extend students’ perspective on logarithmic and exponential functions</td>
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<tr>
<td>(i) $\lim_{x \to 0} (1 + \frac{1}{x})^x = e$</td>
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<td>(ii) $\lim_{h \to 0} (1 + h)^{\frac{1}{h}} = e$</td>
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<tr>
<td>(iii) $\log_a x = \int_{1}^{x} \frac{1}{t} dt$.</td>
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<td>(Please note that the third one is optional which could be taken as an alternative definition for $\log_e x$ or written as $\ln x$).</td>
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<tr>
<td>The general functional properties of logarithmic and exponential functions should also be mentioned.</td>
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<tr>
<td>Logarithmic function $f(x) = \log_a x$ with $a &gt; 0$, $a \neq 1$.</td>
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<tr>
<td>(i) $f(x) + f(y) \approx f(xy)$</td>
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<tr>
<td>(ii) $f(x) - f(y) \approx f(x/y)$</td>
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<tr>
<td>(iii) $f(x^n) = nf(x)$ and Exponential function $g(x) = a^x$ with $a &gt; 0$</td>
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<tr>
<td>(i) $g(x + y) = g(x) \cdot g(y)$</td>
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<tr>
<td>(ii) $g(x - y) = \frac{g(x)}{g(y)}$</td>
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<td>(iii) $g(mx) = (g(x))^m$</td>
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<tr>
<td>And, in particular, the importance of the functions $e^x$ and $\ln x$ in the study of mathematics should be emphasized.</td>
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Unit A3: Mathematical Induction

**Objective:**
1. To understand the Principle of Mathematical Induction.
2. To apply the Principle of Mathematical Induction to prove propositions involving integers.
3. To be able to modify the Principle of Mathematical Induction to suit different purposes.

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| 3.1 The Principle of Mathematical Induction and its applications | 6          | As an introduction, students may be asked to guess the formula for the sum of the first n odd positive integers by considering:

\[
\begin{align*}
1 &= 1 \\
1 + 3 &= 4 \\
1 + 3 + 5 &= 9 \\
&\vdots \\
\end{align*}
\]

After the proposition \(1 + 3 + 5 + \ldots + (2n - 1) = n^2\) is established, students should be led to understand that they should not claim this result is true by considering only a finite number of cases. An illustration of the use of mathematical induction should then follow.

The Principle of Mathematical Induction should be formally written on the board. Teachers may find it easier to explain the Principle by referring to a game of dominoes:

```
1  2  3  4  5
```

Examples should be done on the applications to the summation of series, divisibility and proving inequalities. The Principle of Mathematical Induction may sometimes modified to suit different cases. Examples should also be used to illustrate that both conditions of the Principle must be satisfied to prove a proposition. Further applications include the proofs of:

(i) the binomial theorem for positive integral indices
(ii) De Moivre's theorem for a positive integer \(n\)
(iii) some propositions involving determinants and square matrices
(iv) Leibniz's Theorem and some propositions involving the nth derivative.

As further development, teachers may discuss with the students cases where the Principle has to be modified.

Example:
\[
x^n + y^n \text{ is divisible by } x + y \text{ for all positive odd integers } n.
\]

Example:
The Fibonacci sequence is defined as follows:
\[
a_0 = 0, a_1 = 1 \\
a_{n+1} = a_{n-1} + a_n \text{ for all natural numbers } n.
\]
Prove that
\[
a_n = \frac{1}{\sqrt{5}} \left( \left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n \right) \text{ for all } n
\]

Teachers should point out that a variation of the Principle is required for the proof of these examples. A few more examples on sequences defined by recurrence relations may be discussed.
# Unit A4: Inequalities

**Objectives:**
1. To learn the elementary properties of inequalities
2. To prove simple absolute inequalities
3. To solve simple conditional inequalities

## Detailed Content

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| **4.1 Absolute inequalities** | 6          | Students are expected to use the symbols $a > b$ and $a \geq b$ correctly. Elementary properties of inequalities including:  
(i) For any real number $x$, $x^2 \geq 0$  
(ii) If $a > b > 0$ and $n$ is a positive integer, then $a^n > b^n$  
(iii) If $a > b > 0$ and $x > y > 0$, then $ax > by$ should be revised. Formal proofs of these properties are not required. However, students are expected to be able to deduce simple absolute inequalities from the elementary properties. The following techniques in proving absolute inequalities should be emphasized:  

**Example**  
Prove that $E_1 \geq E_2$  
Proof:  
$E_1 - E_2 = = \geq 0$  
$E_1 \geq E_2$. |
| **4.2 A.M. $\geq$ G.M.** | 4          | The proof of A.M. $\geq$ G.M. may be provided up to four variables in the first instance and the general proof need not be emphasized. (If required, teachers may apply backward induction). Students are expected to apply this result to $n$ variables. |
| **4.3 Cauchy—Schwarz’s inequality** | 3          | Students are expected to understand that the necessary and sufficient conditions for the quadratic form $ax^2 + bx + c$ to be positive for all real values of $x$ are $a > 0$ and $b^2 - 4ac < 0$. Students should be able to apply this result to problems such as finding the range of $C$ for which the expression $Cx^2 + 4x + C + 3$ is positive for all values of $x$. The above result may be used to prove the Cauchy—Schwarz’s inequality. |

## Notes on Teaching

A geometric interpretation of the Cauchy—Schwarz’s inequality for $n = 2$ may be given by using points on the coordinate plane:  
\[
\cos \theta = \frac{OA^2 + OB^2 - AB^2}{2OA \cdot OB} = \frac{(a_1^2 + a_2^2) + (b_1^2 + b_2^2) - (a_1 - b_1)^2 + (a_2 - b_2)^2}{2\sqrt{(a_1^2 + a_2^2)}\sqrt{(b_1^2 + b_2^2)}}
\]
\[
= \frac{a_1b_1 + a_2b_2}{\sqrt{a_1^2 + a_2^2}\sqrt{b_1^2 + b_2^2}}
\]
\[
\cos^2 \theta = \frac{(a_1b_1 + a_2b_2)^2}{(a_1^2 + a_2^2)(b_1^2 + b_2^2)}
\]

Since $\cos^2 \theta \leq 1$,  
\[
(a_1b_1 + a_2b_2)^2 \leq (a_1^2 + a_2^2)(b_1^2 + b_2^2)
\]
Students are also expected to apply the Cauchy—Schwarz’s inequality in solving simple problems.

The concepts of intervals on the real number line should be revised. The definition and properties of the absolute value of a real number should be discussed. Students should be able to solve linear inequalities, quadratic inequalities and inequalities of higher degrees in $x$. Solutions of inequalities involving absolute values such as $|ax^2 + bx + c| \geq d, |x-a| + |x-b| \geq c$ and $(x-a)(x-b) \geq c$ are required. Teachers should also discuss with students inequalities of the form $\frac{P(x)}{Q(x)} \geq 0$, where $P(x)$ and $Q(x)$ are polynomials in $x$. Compound inequalities of the above inequalities should also be taught.

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<td><strong>4.4 Conditional inequalities</strong></td>
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### Unit A5: The Binomial Theorem for Positive Integral Indices

**Objective:**
1. To learn and apply the binomial theorem for positive integral indices.
2. To study the simple properties of the binomial coefficients.

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<tr>
<td>5.1 The binomial theorem for positive integral indices</td>
<td>3</td>
<td>Students should learn how to evaluate n! and ( \binom{n}{r} ). The binomial theorem for positive integral indices may be proved by the Principle of Mathematical Induction. Discussions concerning the notation ( \binom{n}{r} ) should be related to its use as a binomial coefficient. The Pascal's triangle in relation to the coefficients ( \binom{n}{r} ) in the binomial expansion may be discussed. Students are not expected to know the general binomial theorem.</td>
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<tr>
<td>5.2 Application of the binomial theorem for positive integral indices</td>
<td>5</td>
<td>Students should be able to expand expressions using the binomial theorem for positive integral indices. The determination of a particular term or a particular coefficient in a binomial expansion should also be taught. Students are expected to be able to find the greatest term and the greatest coefficient in a binomial expansion. Applications to numerical approximation should be discussed.</td>
</tr>
<tr>
<td>5.3 Simple properties of the binomial coefficients</td>
<td>5</td>
<td>Students should know that both the notations ( \binom{n}{r} ) and ( \binom{r}{n} ) may be used to represent the binomial coefficients. Discussions should include simple properties of the binomial coefficients and the relations between these coefficients such as ( \binom{n}{r} + \binom{n}{r+1} + \binom{n}{r+2} + \ldots + \binom{n}{n} = 2^n; ) ( \binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \ldots + \binom{n}{n}^2 = \frac{(2n)!}{(n!)^2} ) and similar relations.</td>
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N.B. Permutation and combination may be used to introduce the binomial theorem, but problems concerning permutation and combination are not required. Problems involving the use of differentiation and integration may be taught after students have learnt calculus.

### Unit A6: Polynomials and Equations

**Objective:**
1. To learn the properties of polynomials with real coefficients in one variable.
2. To learn division algorithm, remainder theorem and Euclidean algorithm and their applications.
3. To resolve rational functions into partial fractions.
4. To learn the properties of roots of polynomial equations with real coefficients in one variable.

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<tr>
<td>6.1 Polynomials with real coefficients in one variable</td>
<td>5</td>
<td>Students are expected to know the general form of a polynomial with real coefficients in one variable and the following terms: degree of a non-zero polynomial, leading coefficient, constant term, monic polynomial, null or zero polynomial. Definitions of equality, sum, difference and product of two polynomials should also be studied. Also from definition, it is clear that for non-zero polynomials ( f(x), g(x) ) ( \deg (f(x) g(x)) = \deg f(x) + \deg g(x) ) and ( \deg (f(x) + g(x)) \leq \max (\deg f(x), \deg g(x)) ). The greatest common divisor (G.C.D.) or highest common factor (H.C.F.) of two non-zero polynomials should be defined. Students should clearly distinguish between division algorithm and Euclidean algorithm. By the division algorithm, the remainder theorem can be proved. Since students have studied the remainder theorem in lower forms, more difficult problems on this theorem can be given. The Euclidean algorithm is a method of finding the G.C.D. of two polynomials. Some problems on finding the G.C.D. of two polynomials should be given as exercise.</td>
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<td>6.2 Rational functions</td>
<td>4</td>
<td>A rational function should be defined first. Students may come across partial fractions the first time. Teacher may quote a simple example such as ( \frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1} ). The fractions ( \frac{1}{x} ) and ( \frac{1}{x+1} ) are called partial fractions. The rules for resolving a proper rational function into partial fractions should be clearly stated and examples studied. It should be emphasized and illustrated by examples that if a given rational function is improper, it should first be expressed as the sum of a polynomial and a proper fraction. Applications of partial fractions should be studied.</td>
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6.3 Polynomial equations with real coefficients in one variable

Examples:
1. Express \( \frac{x^2 - 3x + 9}{(x - 1)(x - 2)} \) in partial fractions.
2. Resolve \( \frac{x^3}{(x - a)(x^2 + a^2)} \) into partial fractions.
3. Evaluate \( \frac{1}{n!} \) for \( f(t + 1) \)

For the quadratic equation \( ax^2 + bx + c = 0 \) (\( a \neq 0 \)) with \( \alpha, \beta \) as roots, students should be familiar with the relations:
\[
\alpha + \beta = -\frac{b}{a}, \quad \alpha\beta = \frac{c}{a}
\]

For the general polynomial equation of degree \( n \),
\[
f(x) = a_nx^n + a_{n-1}x^{n-1} + ... + ax + a_0 = 0
\]

or \( \sum_{i=0}^{n} a_i x^i = 0 \), the following theorem gives the relations between coefficients and roots:

If \( \alpha_1, \alpha_2, ..., \alpha_n \) are the roots of the polynomial equation \( f(x) = 0 \), then the sum \( s_k \) of all possible products of the \( \alpha_i \)'s taken \( k \) at a time \( (k = 1, 2, ..., n) \) is equal to
\[
(-1)^k \frac{\bar{a}_k}{a_n}
\]

\( \bar{a}_k = a_1a_2a_3...a_k \)

The following properties should be studied in detail:

(i) The number of distinct zeros of a non-zero polynomial is less than or equal to the degree of the polynomial.

(ii) If the polynomial equation with integral coefficients has a rational root of the form \( \frac{p}{q} \) where \( p \) and \( q \) are coprime integers, then \( p \) is an exact divisor of the constant term and \( q \) is an exact divisor of the leading coefficient of the polynomial.

(iii) The condition for repeated (multiple) roots:
For the polynomial equation \( f(x) = 0 \) to have \( x = \alpha \) as a repeated root, it is necessary and sufficient that \( f(\alpha) = 0 \) and \( \alpha \) is also a root of the equation \( f'(x) = 0 \) where \( f'(x) \) denotes the derivative of \( f(x) \).

The more general form will be:
For any positive integer \( k \), \( \alpha \) is a root of multiplicity \( k + 1 \) of the equation \( f(x) = 0 \) if and only if \( f(\alpha) = 0 \) and \( \alpha \) is a root of multiplicity \( k \) of \( f'(x) = 0 \).

and
\[
f(\alpha) = 0, \quad f'(\alpha) = 0, \quad \ldots \quad f^{(k)}(\alpha) = 0
\]

but not of \( f^{(k+1)}(x) = 0 \)

N.B. Complex roots occurring in conjugate pairs will be treated in the study of complex number in Unit A10.
Unit A7: Vectors in \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \)

**Objective:**
1. To study the operations of vectors in \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \).
2. To understand the concept of linearly dependent vectors and linearly independent vectors.
3. To apply vectors in geometrical problems.

**Detailed Content**

<table>
<thead>
<tr>
<th>Detailed Content</th>
<th>Time Ratio</th>
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<tbody>
<tr>
<td>7.1 Definition of Vectors and scalars</td>
<td>1</td>
<td>To begin with this unit, the difference between vectors and scalars should be explained to students. The representation of a vector, both pictorial and written, should be introduced. The current notations of vectors (such as ( \overrightarrow{AB} ), ( \overrightarrow{AB} ), ( \overrightarrow{a} ), ( a )) and their magnitudes (such as (</td>
</tr>
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</table>

**7.2 Operations of vectors**

<table>
<thead>
<tr>
<th>Detailed Content</th>
<th>Time Ratio</th>
<th>Notes on Teaching</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i) Triangle law</td>
<td></td>
<td>Student should know the laws of vector addition (namely, the triangle law, the parallelogram law, and the polygon law), the subtraction of vectors and the multiplication of a vector by a scalar.</td>
</tr>
</tbody>
</table>

- **Triangle law**

\[
\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC} \\
\text{or} \\
\overrightarrow{a} + \overrightarrow{b} = \overrightarrow{c}
\]

It should be pointed out to the student that, when using the law to find \( \overrightarrow{a} + \overrightarrow{b} \), the end point of vector \( \overrightarrow{a} \) must coincide with the initial point of vector \( \overrightarrow{b} \). It should be noted that the validity of the law still holds when \( A, B, C \) are collinear points.

- **Parallelogram law**

\[
\overrightarrow{AB} + \overrightarrow{AC} = \overrightarrow{AD} \\
\text{or} \\
\overrightarrow{a} + \overrightarrow{b} = \overrightarrow{c}
\]

In a similar manner, teachers should remind the students that the initial points of vectors \( \overrightarrow{a} \) and \( \overrightarrow{b} \) must be coincident and in either of the above cases, \( \overrightarrow{c} \) can also be regarded as the resultant of \( \overrightarrow{a} \) and \( \overrightarrow{b} \). The equivalence of the triangle law and the parallelogram law is worth discussing.

- **Polygon law**

\[
\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} + \overrightarrow{EF} = \overrightarrow{AF}
\]

The laws of the vector algebra like commutative law, associative law and distributive law should also be made known to students. The following diagrams may be useful in illustrating these properties.
Notes on Teaching

(a) Commutative law of addition: \( \vec{a} + \vec{b} = \vec{b} + \vec{a} \)

(b) Associative law of addition: \((\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})\)

(c) Associative law for scalar multiplication:

\[(a \beta) \vec{a} = a (\beta \vec{a})\]

Distributive laws for scalar multiplication:

\[a (\vec{a} + \vec{b}) = a \vec{a} + a \vec{b}\]

\[(\alpha + \beta) \vec{a} = \alpha \vec{a} + \beta \vec{a}\]

After understanding the concept of scalar multiplication, students should have no difficulty to deduce the result that if \(\vec{a}\) and \(\vec{b}\) are non-zero vectors such that \(\vec{a} = a \vec{b}\) for some scalar \(a\), then \(\vec{a} \parallel \vec{b}\).

It should be made clear to students concerning the resolution of a vector into component vectors, and the specification of a vectors as a sum of component vectors in \(\mathbb{R}^2\) and \(\mathbb{R}^3\). The resolution of vectors in \(\mathbb{R}^2\) can be introduced with the following examples. In the first example, \(\vec{r}\) is resolved into two components \(\vec{a}\) and \(\vec{b}\) in the directions of \(\vec{a}\) and \(\vec{b}\) respectively. This can be generalized to \(\vec{r} = \alpha \vec{a} + \beta \vec{b}\) where \(\vec{a}\) and \(\vec{b}\) are non-collinear vectors in \(\mathbb{R}^2\) and \(\vec{r} = \alpha \vec{a} + \beta \vec{b} + \gamma \vec{c}\) where \(\vec{a}\), \(\vec{b}\) and \(\vec{c}\) are non-coplanar vectors in \(\mathbb{R}^3\), for scalars \(\alpha\), \(\beta\), and \(\gamma\).

Examples:

1.

2.

Furthermore, scalar multiplication, addition and subtraction of vectors in terms of component vectors should be discussed.
7.3 Resolution of vectors in the rectangular coordinate system

The face that \( \mathbf{i}, \mathbf{j}, \mathbf{k} \) represent the unit vectors in the directions of the positive \( x, y, z \)-axes respectively and that any vector in \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \) can be expressed in the form \( a\mathbf{i} + b\mathbf{j} + c\mathbf{k} \) should be explained in detail.

Students are required to be familiar with the following properties of vectors in terms of \( \mathbf{i}, \mathbf{j} \) and \( \mathbf{k} \):

(i) \[ |a\mathbf{i} + b\mathbf{j} + c\mathbf{k}| = \sqrt{a^2 + b^2 + c^2} \]

(ii) two vectors \( \mathbf{v} = a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k} \) and \( \mathbf{w} = a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k} \) are parallel if \( a_1:b_1:c_1 = a_2:b_2:c_2 \).

Moreover the meaning of direction ratio, direction cosines and direction angle of \( \mathbf{v} \) should be explained with the help of diagrams, and the following properties should be discussed:

(i) \[ \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1 \]

(ii) \[ \dfrac{\mathbf{v}}{||\mathbf{v}||} = \cos \alpha \mathbf{i} + \cos \beta \mathbf{j} + \cos \gamma \mathbf{k} \]


7.4 Linear combination of vectors

The following definitions should be taught:

(i) Let \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n \) be a set of vectors. An expression of the form \( \lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + \lambda_3\mathbf{v}_3 + \ldots + \lambda_n\mathbf{v}_n \), where \( \lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n \) are scalars, is called a linear combination of the vectors \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n \). If the scalars \( \lambda \)'s are not all zero, it is called a non-trivial linear combination, otherwise it is a trivial linear combination.

(ii) A set of vectors \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n \) is said to be linearly dependent if there exists a non-trivial linear combination of them equal to the zero vector, i.e.

\[ \lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + \lambda_3\mathbf{v}_3 + \ldots + \lambda_n\mathbf{v}_n = \mathbf{0} \] where \( \lambda_i \not= 0 \) for some \( i \in \{1, 2, 3, \ldots, n\} \).

(iii) A set of vectors \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n \) is said to be linearly independent if the only linear combination of them equal to zero is the trivial one, i.e.

If \( \lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + \lambda_3\mathbf{v}_3 + \ldots + \lambda_n\mathbf{v}_n = \mathbf{0} \) then \( \lambda_1 = \ldots = \lambda_n = 0 \).

Students should be helped to deduce an immediate result from (ii) that the set of vectors \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n \) is linearly dependent if and only if one of the vectors is a linear combination of the others in the set.

The geometrical significance of linear dependence of vectors in \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \) like the following should be elaborated.

(i) vectors \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) of \( \mathbb{R}^2 \) are linearly dependent if and only if they are parallel;

(ii) vectors \( \mathbf{v}_1, \mathbf{v}_2 \) and \( \mathbf{v}_3 \) of \( \mathbb{R}^3 \) are linearly dependent if and only if they are coplanar.

The definition of the scalar product of two vectors \( \mathbf{a} \) and \( \mathbf{b} \), written as \( \mathbf{a} \cdot \mathbf{b} \), in its usual context that \( \mathbf{a} \cdot \mathbf{b} = ||\mathbf{a}|| ||\mathbf{b}|| \cos \theta \) where \( \theta \) is the angle between \( \mathbf{a} \) and \( \mathbf{b} \), should be taught and the following properties discussed.

1. commutative law of scalar product: \( \mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a} \)

2. distributive law of scalar product: \( \mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} \)

3. \( \mathbf{a} \cdot \mathbf{a} = ||\mathbf{a}||^2 \)

4. two non-zero vectors \( \mathbf{a} \) and \( \mathbf{b} \) are orthogonal if and only if \( \mathbf{a} \cdot \mathbf{b} = 0 \)

5. \( \cos \theta = \dfrac{\mathbf{a} \cdot \mathbf{b}}{||\mathbf{a}|| ||\mathbf{b}||} \)
In particular, teachers should point out that, with references to the fifth property listed above, the scalar product can be used to find the angle between two vectors expressed in Cartesian components. In this connection students may be hinted to show that \( \mathbf{i} \cdot \mathbf{i} = 1 \), \( \mathbf{j} \cdot \mathbf{j} = 1 \), \( \mathbf{k} \cdot \mathbf{k} = 1 \), \( \mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{i} \cdot \mathbf{k} = 0 \), and the result that for \( \mathbf{r}_1 = a_1 \mathbf{i} + b_1 \mathbf{j} + c_1 \mathbf{k} \), \( \mathbf{r}_2 = a_2 \mathbf{i} + b_2 \mathbf{j} + c_2 \mathbf{k} \),

\[
    \mathbf{r}_1 \cdot \mathbf{r}_2 = a_1 a_2 + b_1 b_2 + c_1 c_2.
\]

As for vectors product, the definition must be clearly provided. Special attention should be directed to the proper orientation of the right-hand system.

The vector product of two vectors \( \mathbf{a} \) and \( \mathbf{b} \), written as \( \mathbf{a} \times \mathbf{b} \) is defined as \( \mathbf{a} \times \mathbf{b} = \left| \begin{array}{c}
    \mathbf{i} \\
    \mathbf{j} \\
    \mathbf{k}
  \end{array} \right| \sin \theta \mathbf{e}
\]

where

(i) \( \mathbf{e} \) is the unit vector perpendicular to both \( \mathbf{a} \) and \( \mathbf{b} \); (ii) \( \theta \) is the angle from \( \mathbf{a} \) to \( \mathbf{b} \) measured in the direction determined by \( \mathbf{e} \) according to the right-hand rule.

\[
\mathbf{a} \times \mathbf{b} = \left| \mathbf{i} \right| \left| \mathbf{b} \right| \sin \theta \mathbf{e}
\]

Discussion on the following properties is essential.

(i) \( \mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a} \)

(ii) \( \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b} (\mathbf{a} \cdot \mathbf{c}) - \mathbf{c} (\mathbf{a} \cdot \mathbf{b}) \) and

\[
(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \mathbf{a} (\mathbf{b} \cdot \mathbf{c}) - \mathbf{b} (\mathbf{a} \cdot \mathbf{c})
\]

(iii) \( \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c}) \mathbf{b} - (\mathbf{a} \cdot \mathbf{b}) \mathbf{c} \)

(iv) \( \left| \mathbf{a} \times \mathbf{b} \right|^2 = \left| \mathbf{a} \right|^2 \left| \mathbf{b} \right|^2 - (\mathbf{a} \cdot \mathbf{b})^2 \)

Students may be required to work out for themselves the results like \( \mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0 \) etc. as a prelude to deduce the result that for vectors expressed in Cartesian components \( \mathbf{r}_1 = a_1 \mathbf{i} + b_1 \mathbf{j} + c_1 \mathbf{k} \) and \( \mathbf{r}_2 = a_2 \mathbf{i} + b_2 \mathbf{j} + c_2 \mathbf{k} \),

\[
\mathbf{r}_1 \times \mathbf{r}_2 = (c_1 b_2 - b_1 c_2) \mathbf{i} + (b_1 c_2 - c_1 b_2) \mathbf{j} + (c_1 b_2 - b_1 c_2) \mathbf{k}
\]

Moreover, the properties that

(i) two non-zero vectors \( \mathbf{a} \) and \( \mathbf{b} \) are parallel if and only if \( \mathbf{a} \times \mathbf{b} = 0 \)

(ii) \( \left| \mathbf{a} \times \mathbf{b} \right| \) may be interpreted as the area of the parallelogram formed by the vectors \( \mathbf{a} \) and \( \mathbf{b} \).

are helpful in reinforcing students' mastery of the concept.

Teachers are advised to provide students with detailed explanation and adequate discussion as well as exemplification on the use of relative vectors including position vector and displacement vector. The usual convention that the position vectors of points P and Q with respect to a reference point O are denoted by \( \mathbf{OP}, \mathbf{OQ} \) or \( \mathbf{p}, \mathbf{q} \) respectively and that \( \mathbf{PQ} = \mathbf{q} - \mathbf{p} \) should be highlighted.

The following results should be derived whilst other related generalization are also worth discussing for consolidation.

Position vector of point of division:

Let \( \mathbf{A} \) and \( \mathbf{B} \) be the respective position vectors of \( \mathbf{A} \) and \( \mathbf{B} \) P with reference to the point \( \mathbf{O} \). If \( \mathbf{P} \) divides the line segment \( \mathbf{AB} \) in the ratio of \(\text{m:n} \) then

\[
\mathbf{p} = \frac{m \mathbf{A} + n \mathbf{B}}{m + n}
\]

The different treatments for points of internal and external division should also be discussed. Sometimes the form \( \mathbf{p} = k \mathbf{A} + \mathbf{B} \) should be preferred because, with adequate preparation on the part of the students, this may be interpreted as the vector equation of the straight line passing through \( \mathbf{A} \) and \( \mathbf{B} \). In particular, with respect to the Cartesian system with \( \mathbf{A} \) being the point \( (x_1, y_1, z_1) \), \( \mathbf{B}(x_2, y_2, z_2) \), and \( \mathbf{P}(x, y, z) \), the two-point form of the line

\[
\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}
\]

7.6 Application of vectors in geometry

Indicates the parts deleted
can be easily obtained. At this juncture students may be asked to write down the
direction number of the vector $b-a$ prior to the smooth generalization of the
two-point form into

(i) symmetrical form

$$\frac{x-x_1}{\ell} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$$

(ii) parametric form

$$x = x_1 + \ell t$$
$$y = y_1 + m t$$
$$z = z_1 + n t$$

where $\ell : m : n$ stands for the direction number of the line.

As a continuation, the equation of the plane having normal in the direction $i : m : n$ and passing through $(x_1, y_1, z_1)$ can be introduced as an application of dot product:

$$\langle x-x_1, y-y_1, z-z_1 \rangle = 0$$

In this connection the general equation of a plane $Ax + By + Cz + D = 0$ should be introduced as a supplement with the following properties introduced.

(i) the direction ratios of the normal to the plane are $A : B : C$.

(ii) the perpendicular distance of the point $P(x_1, y_1, z_1)$ to the plane is given by

$$D = \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

(iii) the angle $\theta$ between two planes

$$\cos \theta = \frac{A_1 A_2 + B_1 B_2 + C_1 C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2} \sqrt{A_2^2 + B_2^2 + C_2^2}}$$

(iv) $x_1 \parallel x_2$ if and only if

$$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}$$

$v_1 \perp v_2$ if and only if

$$A_1 v_1 + B_1 v_2 + C_1 c_2 = 0$$

Following the acquisition of the general knowledge of lines and planes, teachers may lead the students to appreciate the fact that

$$A_1 x + B_1 y + C_1 z + D_1 = 0$$

represents the line of intersection of the planes $x_1$ and $x_2$ (if not parallel) and the direction ratios of the line can be found by

$$[B_1, C_1; A_1; A_2, B_2, C_2]$$

Further more the following properties between a line $L$ with direction ratios $p : q : r$ and a plane $Ax + By + Cz + D = 0$ should be discussed

(i) $L \parallel x$ iff $A \times B = C$

(ii) $L \perp x$ iff $A = B = C$

The conditions for two lines to be coplanar should be also studied i.e. two lines are coplanar if and only if they intersect or are parallel.

Suppose $L_1$ is $x = a_1 + p t$, $y = b_1 + q t$, $z = c_1 + n t$

and $L_2$ is $x = a_2 + p_2 t$, $y = b_2 + q_2 t$, $z = c_2 + n_2 t$

$L_1$ and $L_2$ are coplanar if

$$\frac{a_1 - a_2}{b_1 - b_2} = \frac{p_1}{q_1} = \frac{n_1}{n_2}$$

Throughout this sub-unit, teachers are encouraged to apply vector approach as far as possible in deducing the above-mentioned properties or results. In particular the use of dot product to find the projection of a vector $\vec{p}$ along a vector $\vec{r}$ and the use of cross product to evaluate the area of triangle with vertices given should be explained.
### Detailed Content | Time Ratio | Notes on Teaching
---|---|---
#### 8.1 Matrices and their operations | 4 | The general form of a matrix with m rows and n columns, namely an $m \times n$ matrix, should be introduced. Students should know the operations: addition, subtraction, multiplication, and scalar multiplication of matrices and study their properties. The fact that, in general $AB \neq BA$, holds for matrices $A$ and $B$ should be mentioned and explained. Terms like zero matrix, identity matrix, and the transpose of a matrix should be introduced.

The definition of square matrices and their determinants should be defined. The concepts and uses of singular and non-singular matrices should also be made clear to students. Students should be able to evaluate determinants of square matrices and find the inverse of non-singular matrices. They are also expected to have knowledge of simple properties of inverses and determinants like:

- **Properties of inverse**
  - (i) The inverse of a matrix is unique.
  - (ii) A square matrix has inverse if and only if it is non-singular.
  - (iii) If $A$ is non-singular, then $AB = 0$ implies $B = 0$.
  - (iv) If $A$ is non-singular, then $AB = AC$ implies $B = C$.
  - (v) If $A$, $B$ are non-singular, $\lambda$ is a non-zero scalar and $n$ is a positive integer, then $AB, A^T A, \lambda A, A^n$ are non-singular and $(AB)^T = B^T A^T$, $(A^T)^{-1} = A^{-1}$.
  - (vi) The determinant of the product of two square matrices of the same order is equal to the product of the determinants of the matrices, i.e. $\det(AB) = \det A \cdot \det B$ or $|AB| = |A||B|$

**Properties of determinant**

- (i) If two rows (or columns) of a determinant are identical or proportional, the value of the determinant is zero.
- (ii) The interchange of two rows (or columns) changes the sign of the determinant without altering its numerical value.

#### 8.2 Square matrices of order 2 and 3 | 9 | The definition of square matrices and their determinants should be defined. The concepts and uses of singular and non-singular matrices should also be made clear to students. Students should be able to evaluate determinants of square matrices and find the inverse of non-singular matrices. They are also expected to have knowledge of simple properties of inverses and determinants like:

**Properties of inverse**

- (i) The inverse of a matrix is unique.
- (ii) A square matrix has inverse if and only if it is non-singular.
- (iii) If $A$ is non-singular, then $AB = 0$ implies $B = 0$.
- (iv) If $A$ is non-singular, then $AB = AC$ implies $B = C$.
- (v) If $A$, $B$ are non-singular, $\lambda$ is a non-zero scalar and $n$ is a positive integer, then $AB, A^T A, \lambda A, A^n$ are non-singular and $(AB)^T = B^T A^T$, $(A^T)^{-1} = A^{-1}$.
- (vi) The determinant of the product of two square matrices of the same order is equal to the product of the determinants of the matrices, i.e. $\det(AB) = \det A \cdot \det B$ or $|AB| = |A||B|$

**Properties of determinant**

- (i) If two rows (or columns) of a determinant are identical or proportional, the value of the determinant is zero.
- (ii) The interchange of two rows (or columns) changes the sign of the determinant without altering its numerical value.

#### 8.3 Applications to two dimensional geometry | 8 | Students should be familiar with matrix representation of a point and a vector, and, furthermore, reflections, rotations, enlargements, shears, translations, and their compositions. A few examples of such transformations, represented by $2 \times 2$ matrices, are given below:

- **(i)** Reflection in the line $y = (\tan \theta) x$ is given by the matrix $A = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & -\cos 2\theta \end{pmatrix}$.
- **(ii)** Rotation through an angle $\phi$ about the origin is given by the matrix $B = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix}$.
- **(iii)** Enlargement about the origin with scale factor $k > 0$ is given by the matrix $C = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$.
- **(iv)** Shear parallel to $x$-axis with factor $k$ is given by the matrix $D = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$.

For (iii) and (iv), the effect on shape and area should be discussed. It should be made clear to students that under all these transformations of the plane, each point $P(x, y)$ will be transformed to a new point $P'(x', y')$ satisfying $T(x', y') = (x, y)$ where $T$ stands for a transformation.

Elaboration on the composition of the transformations is essential to enable students to have a thorough understanding on matrix multiplication.

**Example**

The effect of reflection in the line $y = (\tan \theta) x$ followed by a rotation through an angle $\theta$ about the origin is given by the product $BA$, complying with the convention $BA(T) = T'$ as mentioned above. It should be emphasized that the product is to be interpreted from right to left. First, apply transformation $A$, then apply transformation $B$.

| indicates the parts deleted |
Other composition of transformations like the following may be mentioned:

Example:
The transformation having equations
\[
\begin{align*}
x' &= y + 2 \\
y' &= -x + 4
\end{align*}
\]
whose representation in matrix form is
\[
\begin{pmatrix}
x' \\
y'
\end{pmatrix} = 
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix} +
\begin{pmatrix}
2 \\
4
\end{pmatrix}
\]
The transformation with matrix \[
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
\]
is a clockwise rotation about the origin through \(90^\circ\). Hence, it could be viewed as a rotation followed by a translation, however, with \((3, 1)\) as the invariant point the transformation can be regarded as a clockwise rotation about \((3, 1)\) through \(90^\circ\).

---

Unit A9: System of Linear Equations in 2 or 3 Unknowns

Objective:
1. To solve a system of linear equations using Gaussian elimination.
2. To recognize the existence and uniqueness of solution.

Detailed Content | Time Ratio | Notes on Teaching
--- | --- | ---
9.1 Gaussian elimination and Echelon form | 5 | A matrix which satisfies the following 2 properties is said to be in Echelon form:
(1) The 1st k rows are non-zero;
the other rows are zero.
(2) The 1st non-zero element in each non-zero row is 1, and it appears in a column to the right of the 1st non-zero element of any preceding row.

Example:
The following 5 x 8 matrix is in Echelon form:
\[
\begin{array}{cccccccc}
0 & 1 & * & * & * & * & * & * \\
0 & 0 & 1 & * & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

Students should be able to solve a system of linear equations in two or three unknowns by using Gaussian elimination, which reduces a matrix in Echelon form by elementary operations on its rows.

Example:
Solve the system
\[
\begin{align*}
x_1 + 2x_2 + x_3 &= 2 \\
3x_1 + x_2 - 2x_3 &= 1 \\
x_3 + 3x_2 - x_3 &= 3 \\
2x_1 + 4x_2 + 2x_3 &= 4
\end{align*}
\]
The augmented matrix
\[
\begin{pmatrix}
1 & 2 & 1 & 2 \\
3 & 1 & -2 & 1 \\
4 & -3 & -1 & 3 \\
2 & 4 & 2 & 4 \\
1 & 2 & 1 & 2 \\
0 & 1 & 1 & 1 \\
0 & -11 & -5 & -5 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}
\]
The original system of equations is equivalent to the system of equations
\[
\begin{align*}
x_1 - x_2 &= 0 \\
x_1 + x_2 + x_3 &= 1 \\
x_3 &= 1
\end{align*}
\]
which gives \(x_1 = 1, x_2 = 0, x_3 = 1\).
9.2 Existence and uniqueness of solution

Students should be able to know the conditions for the existence and uniqueness of solution for a system of linear equations in two or three unknowns.

For a system of linear equations in two unknowns:

(i) If \( \text{det} \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \neq 0 \), the system has unique solution. Geometrically, the equations represent a pair of intersecting straight lines.

(ii) If \( \text{det} \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = 0 \) and \( \text{det} \begin{pmatrix} d_1 & b_1 \\ d_2 & b_2 \end{pmatrix} \neq 0 \), the system has no solution. Geometrically, the equations represent a pair of parallel (but not coincident) straight lines.

(iii) If \( \text{det} \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = 0 \) and \( \text{det} \begin{pmatrix} d_1 & b_1 \\ d_2 & b_2 \end{pmatrix} = 0 \), the system has infinite number of solutions. Geometrically, the equations represent a pair of coincident straight lines.

For systems of equation in three unknowns, examples like the following should be mentioned. The corresponding geometrical meaning may be discussed if the students have grasped some ideas of three dimensional coordinate geometry.

(i) In solving the equations

\[
\begin{align*}
2x + y - z &= 7 \\
5x - 4y + 2z &= 1 \\
7x - 3y + 6z &= 8
\end{align*}
\]

it is obvious that the third one is redundant. Teachers may discuss with the students on the method to obtain the solution

\[
x = \frac{20 - 32}{13}, \quad y = \frac{33 + 19}{13}, \quad z = \lambda,
\]

where \( \lambda \) is arbitrary.

(ii) Solving equations like

\[
\begin{align*}
x + y + z &= 3 \\
2x - 3y + 2z &= 1 \\
3x - 2y + 3z &= 7
\end{align*}
\]

which are inconsistent.

Following this manner, the conditions for the existence and uniqueness of solution for a system of equations in three unknowns may be given in more abstract terms.

---

### Unit A10: Complex Numbers

**Objective:**

1. To learn the properties of complex numbers, their geometrical representations and applications.
2. To learn the De Moivre's Theorem and its applications in finding the nth roots of complex numbers, in solving polynomial equations and proving trigonometric identities.

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| 10.1 Definition of complex numbers and their arithmetic operations | 3 | A short introduction of the symbol \( i \) should be given. The number \( z = x + yi \), where \( x, y \) are real, is called a complex number and \( x \) and \( y \) are known respectively as its real (\( \text{Re} \) z) and imaginary (\( \text{Im} \) z) parts. When \( x = 0 \), \( y \neq 0 \), \( z = yi \) is said to be purely imaginary and when \( y = 0 \), \( z = x \) is real. Students may be asked what definition should be adopted for the equality of complex numbers, however there is no ordering property for complex numbers. The sum, difference, product and quotient of two complex numbers should be defined. Students are expected to know the definitions of the terms modulus \( |z| \), argument \( \arg z \), principal (value of) argument (or amplitude) and conjugate \( \overline{z} \) of a complex number \( z \). The complex number \( z = r(\cos \theta + i \sin \theta) \), in the modulus --- argument form (polar form), can be written as \( z = r \cos \theta \). Students are expected to know the following properties of complex numbers:

(i) \( |z_1z_2| = |z_1||z_2| \)

(ii) \( \arg z_1z_2 = \arg z_1 + \arg z_2 + 2k\pi \) where \( k \) is an integer

(iii) \( \frac{z_1}{z_2} = \frac{|z_1|}{|z_2|} \)

(iv) \( \frac{z_1}{z_2} = \frac{\arg z_1 - \arg z_2 + 2k\pi}{z_2} \) where \( k \) is an integer and \( z_2 \neq 0 \).

Properties about conjugate complex numbers should be taught:

1. \( \overline{z} = z \)
2. \( \overline{z} = 0 \) if \( z = 0 \)
3. A complex number is self-conjugate (conjugate to itself) if it is real.
4. \( z - \overline{z} = i|z|^2 \) |
| 10.2 Argand diagram, argument and conjugate | 6 | |

[Notes on Teaching: Indicates the parts deleted]
10.3 Simple applications in plane geometry

5

Students are expected to know the following inequalities:
(i) \(|\text{Re}z| \leq |z|
(ii) |\text{Im}z| \leq |z|
(iii) |z_1 + z_2| \leq |z_1| + |z_2| (Triangle Inequality)

The geometrical representation of complex numbers in an Argand diagram should be studied. Students should know the terms real axis and imaginary axis. The representation of a complex number in polar form and its geometrical meaning should also be taught.

The notation \(e^{i\theta}\) for \(\cos \theta + i\sin \theta\) may be introduced so that \(z = re^{i\theta}\). The notation is known as the exponential form or the Euler form of a complex number.

Students are expected to know the geometrical meaning of the triangle inequality. Various uses of complex number in plane geometry should be studied. The following are two examples:

1. In the Argand diagram, \(XYZ\) is an equilateral triangle whose circumcentre is at the origin. If \(X\) represents the complex number \(1 + i\), find the numbers represented by \(Y\) and \(Z\).

2. If \(z_1, z_2\) and \(z_3\) are three distinct complex numbers denoting the vertices of an equilateral triangle, then

\[ z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1.\]

Examples on the loci of points moving on the Argand plane should be studied. Two simple examples are given below:

(i) To find the locus of the point \(z\) such that \(|z - a| = k\), where \(a\) is a complex number and \(k\) is a positive constant.

(ii) To find the locus of the point \(z\) which moves such that \(|z - a| = k\), where \(a, b\) are complex numbers, for various values of the positive constant \(k\).
### 10.4c $n^{th}$ roots of a complex number and their geometrical interpretation

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<td>$n^{th}$ roots of a complex number and their geometrical interpretation</td>
<td>6-4</td>
<td>can be used to express powers of $\cos \theta$ and $\sin \theta$ in terms of sines and cosines of multiples of $6$. For example, students should be able to express $\cos^2 \theta \sin^4 \theta$ as a sum of sines of multiples of $6$ and $\cos^3 \theta \sin^6 \theta$ as a sum of cosines of multiples of $6$.</td>
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Students should learn the meaning of the $n^{th}$ roots of a complex number. The $n^{th}$ roots of unity should be studied in detail.

Several examples can be discussed in class:
1. To find the fifth roots of $-1$.
2. To solve the equation $z^4 + z^3 + z^2 + z + 1 = 0$.
3. To find the cube roots of $1 + i$.
4. Factorize $z^{15} - 2z^{10} \cos \theta + 1$ into real quadratic factors.

---

### Unit B1: Sequence, Series and their Limits

**Objective:**
1. To learn the concept of sequence and series.
2. To understand the intuitive concept of the limit of sequence and series.
3. To understand the behaviour of infinite sequence and series.

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| 1.1 Sequence and series | 6 | Clear concepts of sequence and series should be provided. The following suggested versions may be adopted:

If $a_n$ is a function of $n$ which is defined for all positive integral values of $n$, its values $a_1, a_2, a_3, ..., a_n, ...$ are said to form a sequence. The sequence is finite or infinite according to the numbers of terms of it being finite or infinite. Furthermore $a_1 + a_2 + ... + a_n, ...$ is said to form a series. Likewise, it is finite or infinite according to the numbers of terms contained. The notation

$$S_n = \sum_{r=1}^{n} a_r$$

is commonly used.

Some simple rules concerning the operations of sequences and series may be introduced. For the sake of convenience, denote the sequences $a_1, a_2, a_3, ...$ and $b_1, b_2, b_3, ...$ by $\{a\}$ and $\{b\}$, then

1. $\{a\} \pm \{b\} = \{a \pm b\}$
2. $\{a\} \cdot \{b\} = \{a \cdot b\}$

$\{a\} \cdot \{b\}$ implies the idea of termwise operations may be touched upon.

Regarding series, the following methods of summation should be discussed.

1. **Mathematical induction: already dealt with in Unit A3.**
2. **Method of difference:** teachers should amplify in the expressing the $r$th term of the series as the difference of $f(r + 1)$ and $f(r)$ where $f(x)$ is a function of $x$. i.e. if $a_r = f(r + 1) - f(r)$

$$\sum_{r=1}^{n} a_r = \sum_{r=1}^{n} (f(r + 1) - f(r))$$

$$= f(n + 1) - f(1).$$

Some typical examples are $\sum_{r=1}^{n} \frac{1}{r(r+1)}$ and $\sum_{r=1}^{n} (r+1)$.
1.2 Limit of a sequence and series

Detailed Content

- For series whose terms are presented in a recurrence of the form \( a_n = a_{n-1} + f(n) \) or \( a_n = a_{n-1} + b_{n-1} \), some basic methods should be introduced, especially for the latter one. The following approach may be discussed:

  Suppose \( a, b \) are the roots of the auxiliary equation \( x^2 = Ax + B \).
  1. if \( \alpha \neq \beta, a_n = k_1 \alpha^n + k_2 \beta^n \)
  2. if \( \alpha = \beta, a_n = (k_1 + k_2 n) \alpha^n \)

  where \( k_1 \) and \( k_2 \) are constants to be determined.

- The concept of the limit of a sequence should be taught with an intuitive approach. The following version may be considered:

  Let \( a_1, a_2, a_3, \ldots, a_n, \ldots \) be a sequence. If for all sufficiently large values of \( n \), the difference between \( a_n \) and a constant \( \ell \) is as small as we please, we say that \( a_n \to \ell \) when \( n \to \infty \) or \( \lim a_n = \ell \).

  Teachers should emphasize on the following points:
  1. \( \ell \) is called the limit of the sequence;
  2. the limit \( \ell \), if exists, is unique;
  3. the sequence is said to converge to \( \ell \) or the sequence is convergent with limit \( \ell ; \)
  4. if a sequence does not converge to any limit, it is said to be divergent.

  Some common properties of convergent sequences should be included in the discussion with students:

  (i) \( \lim a_n = 1 \) for \( a > 0 \).
  (ii) \( \lim a_n = 0 \) for \( a < 0 \).
  (iii) \( \lim n a_n = n \lim a_n \).
  (iv) \( \lim a_n + b_n = \lim a_n + \lim b_n \).

N.B. Sequences could be classified as convergent, divergent (to \( +\infty \) or \(-\infty \)) or oscillatory (does not converge nor diverge to \( +\infty \) or \(-\infty \)).

Some common properties of convergent sequences should be included in the discussion with students:

For series whose terms are presented in a recurrence of the form \( a_n = Aa_{n-1} + Ba_{n-2} \), some basic methods should be introduced, especially for the latter one. The following approach may be discussed:

Let \( a_1, a_2, a_3, \ldots, a_n, \ldots \) be a convergent sequence with limits \( a \) and \( b \) respectively. The following sequence are also convergent:

- \( \lambda a_1, \lambda a_2, \lambda a_3, \ldots \) converges \( \lambda a \) where \( \lambda \) is a constant.
- \( a_1 + b_1, a_2 + b_2, a_3 + b_3, \ldots \) converges to \( a + b \).
- \( a_1 b_1, a_2 b_2, a_3 b_3, \ldots \) converges to \( a b \).
- \( a_1 b_1, a_2 b_2, a_3 b_3, \ldots \) converges to \( a b \) provided \( b \neq 0 \).

Finally, students should be led to appreciate the following results that

(i) for the convergent sequence \( a_1, a_2, a_3, \ldots \) with limit \( a \),

\[ \lim a_{n+k} = \lim a_n = a \]

where \( k \) is a positive integer.

(ii) for the two convergent sequences

\[ a_1, a_2, a_3, \ldots \]

and \( b_1, b_2, b_3, \ldots \) converges \( c_1, c_2, c_3, \ldots \) such that \( a_i \leq c_i \leq b_i \) when \( i > k \) for some positive integer \( k \), then \( c_1, c_2, c_3, \ldots \) also converges and to the same limit \( \ell \). This property is commonly known as the Sandwich Theorem. Teachers may also touch upon the meaning of monotonic sequence and bounded sequence to broaden students' understanding.

As for infinite series, a parallel treatment could be provided as follows:

(1) Concept of convergence

The series \( a_1 + a_2 + a_3 + \ldots \) is convergent if \( \lim \sum_{i=1}^{n} a_i = S \) exists and the series is said to be convergent to the limit. (Sometimes \( S \) may be called the sum of the series.) If \( S \), represents \( a_1 + a_2 + \ldots + a_n \), then the result may be stated as \( a_n \to S \) as \( n \to \infty \) or \( \lim S_n = S \). (\( S_n = a_1 + a_2 + \ldots + a_n \) is commonly known as the \( n \)th partial sum). And, in a more or less the same situation, divergent series and/or oscillatory series may be introduced subject to teachers' preference.

(2) Properties of convergent series

\[ a_1 + a_2 + a_3 + \ldots \] with limit \( S \) and

\[ b_1 + b_2 + b_3 + \ldots \] with limit \( S' \) then:

(a) \( a_1 b_1 + a_2 b_2 + a_3 b_3 + \ldots \) converges to \( S S' \) where \( S \) is a constant.
(b) \( (a_1 b_1) + (a_2 b_2) + (a_3 b_3) + \ldots \) converges to \( S + S' \).
(c) If \( a_1 + a_2 + a_3 + \ldots \) is convergent, then \( \lim a_n = 0 \).
1.3 Convergence of a sequence and series

Further properties of convergent sequences like
(i) convergent sequences are bounded
(ii) a monotonic and bounded sequence is convergent should be introduced. Some typical convergent and divergent sequences should be discussed so as to illustrate the method in finding limits of sequences. The following examples may be considered:

(A) Convergent sequences
(i) \( a_n = x^n \) with \(|x| < 1\)
(ii) \( a_n = \frac{1}{n} \)
(iii) \( a_n = \frac{x^n}{n^2} \)

(B) Convergent series
(i) \( 1 + \frac{1}{2} + \frac{1}{3} + \ldots \) with \(|r| < 1\)
(ii) \( 1 + \frac{1}{1} + \frac{1}{1.2} + \frac{1}{1.2.3} + \ldots \)
(iii) \( 1 + \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots \)
(iv) \( 1 + \frac{1}{3} + \frac{1}{3} + \frac{1}{4} + \ldots \)

(C) Divergent series
(i) \( \sum \frac{1}{n} \)
(ii) \( \sum \frac{1}{n} \)
(iii) \( \sum \frac{1}{\sqrt{n}} \)

Some typical applications of the Sandwich Theorem should be included for illustration whereas convergence tests of series are not required.

2.1 Limit of a function

An intuitive understanding of the concept of limit of function is expected. As a matter of fact, the concept of the limit of a function \( y = f(x) \) at the point \( x = a \) can be related to the concept of the limit of a sequence. This is done by allowing the independent variable to run through a convergent sequence of numbers \( \{x_n\} \) tending to the limit \( a \) (the abscissa sequence), and considering the ordinate sequence \( \{f(x_n)\} \). Thus a more vivid visualization of the fact that \( \{f(x_n)\} \) tends to a finite value \( \ell \) as \( \{x_n\} \) tends to a could be established i.e.

\[ f(x) \to \ell \text{ when } x \to a \quad \text{or} \quad \lim_{x \to a} f(x) = \ell. \]

Some teachers may perhaps prefer just to focus students' attention to the fact that the difference between \( f(x) \) and \( \ell \) can be made arbitrarily small when \( x \) is sufficiently close to \( a \) so as to reinforce the idea that \( f(x) \to \ell \) when \( x \to a \). It must be pointed to students that, from the existence of the value \( f(a) \) of the function, one can certainly not conclude that the limit \( \lim_{x \to a} f(x) \) must also exist and be equal to \( f(a) \), though this is very often the case. The following example may be considered:

\[ f(x) = \begin{cases} \frac{1}{x} & \text{when } x \neq 0 \\ 0 & \text{when } x = 0 \end{cases} \]

In which \( f(0) = 0 \) and \( \lim_{x \to 0} f(x) = 1 \).

It may be important in the passage to the limit whether the independent variable approaches the value \( a \) in the sense of increasing values of \( x \), that is, from the left, or in the sense of decreasing values of \( x \), that is from the right. In these cases, the limits are referred to, respectively, as the left-hand limit, usually denoted by \( \lim_{x \to a^-} f(x) \), and the right-hand limit \( \lim_{x \to a^+} f(x) \). In this context, students could be led easily to appreciate that the function \( f(x) \) has a limit as \( x \to a \) if and only if the left-hand and right-hand limits as \( x \to a \) are equal. For a more comprehensive understanding of limit, teachers should touch upon the case when \( x \to \infty \) by reiterating that the difference between \( f(x) \) and \( \ell \) could be made arbitrarily small when \( x \) is sufficiently large. Symbolically, it is presented as \( \lim_{x \to \infty} f(x) = \ell \).
The following properties of the limit of a function should be included in discussion:

(i) \( \lim[f(x) + g(x)] = \lim f(x) + \lim g(x) \)  

(ii) \( \lim[kf(x)] = k \lim f(x) \)  

(iii) \( \lim[f(x)g(x)] = \lim f(x) \cdot \lim g(x) \)  

(iv) \( \lim \frac{f(x)}{g(x)} = \frac{\lim f(x)}{\lim g(x)} \)  

provided \( \lim g(x) \neq 0 \)  

(v) if \( f(x) \leq k(x) \leq g(x) \) holds when \( x \) is close to \( a \) and \( \lim f(x) = \lim g(x) = L \)  

then \( \lim h(x) = L \) also.

Some important limits, like the following, should be introduced:

(1) \( \lim_{x \to 0} \frac{\sin x}{x} = 1 \)

(2) \( \lim_{x \to 0} \frac{1 + x - 1}{x} = 1 \)

(3) \( \lim_{x \to 0} \frac{e^x - 1}{x} = 1 \)

(4) \( \lim_{x \to 0} \frac{\ln (1 + x)}{x} = 1 \)

(5) \( \lim_{x \to 0} \frac{1}{x} = \frac{1}{x} \)

Adequate practice in evaluating the limit of function should be provided for consolidation.

2.2 Continuity of a function

Continuity should be defined on the basis of the limit of function with an intuitive approach; the \( \varepsilon - \delta \) approach may not be desirable. The following suggested version may be considered:

A function \( f(x) \) is continuous at \( x = a \) if \( \lim_{x \to a} f(x) \) exists and is equal to \( f(a) \).

A function is continuous in an interval if it is continuous at every point of the interval.

Some common functions like

(i) \( f(x) = x^2 \) which is continuous in every interval;  

(ii) \( f(x) = \frac{1}{x} \) which is not continuous in the whole interval \( 0 \leq x \leq 5 \)

should be discussed as a prelude to introduce the concept of point of discontinuity.

It should be noted that just informal treatment on this concept is expected, however, teachers are advised to provide students with a good spectrum of examples as a form of reinforcement. Furthermore, the fact that the sum, difference and product of two functions continuous at \( x = a \) are likewise continuous at this point. Their quotient is continuous provided that the denominator is not zero at \( x = a \). Teachers may quote a lot of everywhere continuous functions to initiate students' further study on this topic:

(i) polynomial function \( f(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_0 \)

(ii) exponential function \( f(x) = a^x \), \( a > 0 \)

(iii) logarithmic function \( f(x) = \log_a x \), \( a > 0, a = 1 \)

(iv) trigonometric functions like \( \sin x, \cos x \).

Concerning the continuity of composite function, teachers may consider the suggested version:

Let \( y = f[g(x)] \) be a composite function, when inner function \( g(x) \) is continuous at \( x = a \) and whose outer function \( y = f(t) \) is continuous at \( t = g(a) \), then the composite function \( y = f[g(x)] \) is continuous at \( x = a \).

Also teachers may highlight the fact that every continuous function of a continuous function is again continuous.

Teachers should point out that functions which are continuous in an interval form a class of functions with noteworthy properties, like the following, and the discussion of them is expected but formal proof of them is not desirable.
## Detailed Content

### Time Ratio

### Notes on Teaching

#### 2.3 Differentiability of a function

1. If a function \( f(x) \) is continuous in a closed interval \([a, b]\) with \( f(a) = A \) and 
   \( f(b) = B \) where \( A \neq B \), then \( f(x) \) takes every value between \( A \) and \( B \) at least 
   once. (The Intermediate Value Theorem).

2. A function that is continuous in a closed interval is bounded there.

3. A function that is continuous in a closed interval attains maximum and 
   minimum values in the interval. (Properties (i) and (iii) are known as 
   Weierstrass Theorem).

Regarding the differentiability of a function \( f(x) \) at the point \( x = a \) the following 
version may be considered:

A function \( f(x) \) is said to be differentiable at the point \( x = a \) if and only if the limit 
\[
\lim_{{h \to 0}} \frac{f(x + h) - f(x)}{h}
\]
exists.

Teachers may also at the same time put forth the idea that if a function is 
differentiable at a certain point, it is also continuous there and that continuity is a 
necessary condition for differentiability but not a sufficient one. Moreover, the 
definition of the derivative of a function at \( x = a \), being the value of the above limit, 
can be taught very smoothly following students' acceptance of the idea of 
differentiability. The common notations for the derivative of \( f(x) \) at \( x = a \), like 
\[
\frac{d}{dx} f(x) \\
\text{at } x = a
\]
should be mentioned.

Teachers may also touch upon the differentiability of a function in the whole 
interval in the context that the derivative of the function exists for all points in that 
interval. Furthermore, teachers should elaborate on the property that to each value \( x \) in 
the interval, there corresponds the derivative \( f(x) \) of the function at the point \( x \); thus 
\( f(x) \) is again a function of \( x \) and is called the derived function of \( f(x) \).

At this stage, ample examples should be worked out to reinforce students' mastery 
of the concept and skills concerning differentiation. In particular, examples to find the 
derivative of different typical functions from the first principles are of particular 
importance. In this connection, adequate practices are indispensable. The following 
examples are typical:

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<td>(1) Find the derivative of the functions from the first principles</td>
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<tr>
<td>(i) ( x^2 ) at ( x = 1 )</td>
<td></td>
<td></td>
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<tr>
<td>(ii) ( e^x ) at ( x = 0 )</td>
<td></td>
<td></td>
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<tr>
<td>(iii) ( \sin x ) at ( x = \pi/4 )</td>
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<tr>
<td>(2) Differentiate, from the first principles, the following functions</td>
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<tr>
<td>(c) ( f(x) = x^n ) where ( n ) is a positive integer</td>
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<td></td>
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<tr>
<td>(d) ( f(x) = e^x )</td>
<td>13</td>
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Unit B3: Differentiation

Objective: (1) To acquire different techniques of differentiation.
(2) To learn and acquire techniques to find higher order derivative.
(3) To understand the intuitive concept of Rolle's Theorem and Mean Value Theorem.

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| **3.1 Fundamental rules for differentiation** | 4 | As a continuation, the following rules should be taught:

1. \( \frac{d}{dx}(k) = 0 \), where \( k \) is a constant
2. \( \frac{d}{dx}(x^r) = rx^{r-1} \), where \( r \) is real
3. \( \frac{d}{dx}[(f(x)g(x))] = \frac{d}{dx}f(x)g(x) + f(x)\frac{d}{dx}g(x) \)
4. \( \frac{d}{dx}[kf(x)] = kf'(x) \), where \( k \) is a constant
5. \( \frac{d}{dx} \left[ \frac{f(x)}{g(x)} \right] = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2} \) (quotient rule)

Proofs of the above rules should be mentioned or presented as a form of practice in order to strengthen students' mastery of the concept and skill. From (3) to (6), the existence of the derivatives of \( f(x) \) and \( g(x) \) should be emphasized. Regarding (2), a proof for \( r \) being integral will be enough while for the general case \( r \) being real the proof may be provided at a later stage till the students have learnt the Chain rule. Typical examples in using the above rules to obtain derivative of various common functions should be done for illustration.

| **3.2 Differentiation of trigonometric functions** | 2 | Differentiation of the following functions should be taught:
1. \( \sin x \)
2. \( \cos x \)
3. \( \tan x \)
4. \( \cosec x \)
5. \( \sec x \)
6. \( \cot x \)

Students may be encouraged to do the proof themselves under teachers' supervision and, in particular, they should be reminded to derive the results for (4) to (6) using the quotient rule.

For a composite function \( y = f[g(x)] \), the derivative is obtained through the chain rule:

\[
\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}
\]

For the inverse function \( x = f^{-1}(y) \) of \( y = f(x) \), the derivative is obtained through

\[
\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}
\]

It is suggested that examples like

\[
\frac{d}{dx}[(\sin^{-1} x)], \quad \frac{d}{dx}[(\cos^{-1} x)], \quad \frac{d}{dx}[(\tan^{-1} x)]
\]

\[
\frac{d}{dx}[(x^n)] \quad \text{with } n \text{ being a positive integer}
\]

should be used for illustration.

| **3.3 Differentiation of composite functions and inverse functions** | 4 | It is often necessary to differentiate a function defined implicitly by \( F(x, y) = 0 \). This is done by differentiating both sides of the given equation with respect to the independent variable \( x \) and applying the rules mentioned above. Various illustrating examples should be included to enrich the discussion. The following are some suggestions:

(i) If \( x \cos y^2 + y \sin 2x = 1 \), find \( \frac{dy}{dx} \)

(ii) Given \( 2x^2 - y^2 + 12x - 2y + 3 = 0 \), find \( \frac{dy}{dx} \) at the point (2, 5).

(iii) Find \( \frac{dy}{dx} \) for \( \cos(x^2 - y^2) = xy \).

| **3.4 Differentiation of Implicit functions** | 2 | Indicates the parts deleted

---

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### 3.5 Differentiation of Parametric Equations

A parametric representation of a function $y = f(x)$ is given by $x = u(t)$ and $y = v(t)$. Hence $y$ can be expressed as a composite function of the parameter $t$ in the form $y = f(u(t))$. By applying the chain rule for differentiation, the result:

$$\frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt} \quad \text{and hence} \quad \frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt},$$

or $f'(x) = \frac{v'(t)}{u'(t)}$.

Can be obtained. It should be clarified that in this derivation it is assumed that $u(t)$ and $v(t)$ are differentiable and $u'(t) \neq 0$.

Typical examples for illustration include finding $\frac{dy}{dt}$ for the following functions:

1. The ellipse $x = acost, y = b \sin t$
2. The cycloid, $x = a(t - \sin t), y = a(1 - \cos t)$

The following rules should be taught and their proofs may be provided with the suggested approach.

1. \( \frac{d}{dx} (\ln x) = \frac{1}{x} \) (using $\lim_{t \to 0} (1 + x)^{1/x} = e$)
2. \( \frac{d}{dx} e^x = e^x \) (using $\frac{d}{dx} (e^y) = \frac{1}{y}$ and chain rule, where $y = e^x$, or applying the rule about the derivative of inverse function)
3. \( \frac{d}{dx} (\log_a x) = \frac{1}{x \ln a} \)
4. \( \frac{d}{dx} (a^x) = a^x \ln a \)

Examples provided should include functions of the types like $e^{x^2}$ and $\log_a \sqrt{x^2 + 1}$.

(N.B. At this juncture the proof for the formula \( \frac{d}{dx} x^n = nx^{n-1} \) when $n$ is rational and when $n$ is real may be mentioned for the sake of completeness.)

### 3.6 Differentiation of Logarithmic and Exponential Functions

### 3.7 Higher Order Derivatives and Leibniz's Theorem

Teachers should also highlight some common applications of logarithmic differentiation as follows:

- When $y$ is a complicated function of $x$ and especially when it involves a variable as index, the value of $\frac{dy}{dx}$ may sometimes be more easily obtained by logarithmic differentiation. Typical examples of this kind include functions like $y = x^x$ and $y = (x+a)^{(x+b)}$.

The definition of higher order derivatives and the symbols $f^{(n)}(x)$, $f^{(n)}(x)$, $\frac{d^n y}{dx^n}$ should be introduced. Also the abilities to find higher order derivatives of functions given in parametric form and to apply the Leibniz's Theorem, viz

$$\frac{d^n}{dx^n} (uv) = \sum_{r=0}^{n} \binom{n}{r} u^{(r)} v^{(n-r)}$$

are expected. Students may attempt to prove the theorem by mathematical induction. Examples showing the use of Leibniz's Theorem in obtaining relations involving higher order derivatives especially of implicit functions should be illustrated. Examples of this kind include

1. Find the $n$th derivatives of $\cos^2 x \sin x$ and $x^2 \cos x$.
2. Let $f(x) = \tan^{-1} x$, show that $(1 + x^2) f'(x) + 2x f''(x) = 0$ and hence obtain the $n$th derivative of $f(x)$ for $x = 0$.
3. Given $f(x) = \frac{x^3}{x^2 - 1}$, show that $f^{(n)}(0) = \begin{cases} 0 & \text{if } n \text{ is even} \\ (-1)^n n! & \text{if } n \text{ is odd} \end{cases}$

where $n$ is an integer and $n \geq 3$.

The intuitive concept of the Rolle's Theorem and Mean Value Theorem as well as their geometrical interpretation should be taught. For able students the proof may be mentioned. Simple and straightforward applications of the theorems are expected. The following examples may be considered:

### 3.8 The Rolle's Theorem and Mean Value Theorem

Indicates the parts deleted
1. If $f(x) = 0$ for all $x$ in an interval, then $f(x)$ is constant in that interval.

2. If $f(x) = g'(x)$ for all $x$ in an interval, then $f(x)$ and $g(x)$ differ in that interval by a constant.

3. Prove that if
   
   \[ \frac{a_0}{n+1} + \frac{a_1}{n} + \ldots + \frac{a_n}{2} = 0 \]

   then the equation
   
   \[ a_0x^n + a_1x^{n-1} + \ldots + a_n = 0 \]

   has at least one root between 0 and 1.

Unit B4: Application of Differentiation

Objective: (1) To learn and to use the L'Hopital's Rule.

(2) To learn the applications of differentiation.

### Detailed Content | Time Ratio | Notes on Teaching

| 4.1 The L'Hopital's Rule | 4 |

Limits having the following indeterminate forms should be introduced:

- $0 \cdot \infty$, $0 \cdot -\infty$, $-\infty \cdot -\infty$, $-\infty$, $\infty$, $1^n$

Accompanying examples illustrating the type mentioned are highly desirable. The L'Hopital's Rule

\[ \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \]

for the indeterminate form $\frac{0}{0}$ and $\frac{\infty}{\infty}$ should be taught in the first place.

The examples that follow may be considered:

1. \( \lim_{x \to 0} \frac{\cos^2 x}{e^{2x} - 3x} \)

2. \( \lim_{x \to a} \frac{\sin(x-a)}{2 \tan(x-a)} \)

Teachers should emphasize that $f(x)$ should be simplified before taking limit and the process can be repeated until $\lim_{x \to a} \frac{f^{(n)}(x)}{g^{(n)}(x)}$ is obtained in a non-indeterminate form.

As for the other indeterminate forms, examples should be worked out showing that they can be expressed in the determinate forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$ so that the rule may be applied.

The following examples may be considered:

1. \( \lim_{x \to 0} \frac{\tan x}{\cot x} \)

2. \( \lim_{x \to a} x^a \)

3. \( \lim_{x \to \frac{\pi}{2}} \sin x \tan x \)

The proof of the L'Hopital's Rule is not required.
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<tr>
<th>Detailed Content</th>
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<tbody>
<tr>
<td>4.2 Rate of change</td>
<td>3</td>
<td>The meaning of ( \frac{dy}{dx} ) as the rate of change of ( y ) with respect to ( x ) should be introduced and thoroughly discussed with reference to some common quantities like velocity and acceleration etc.</td>
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<td></td>
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<td>Examples for consideration:</td>
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<tr>
<td>(1) A snowball is melting with its volume decreasing at a constant rate of ( x \text{ cm}^3/\text{s} ). When its radius is ( r \text{ cm} ), find</td>
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<tr>
<td>(a) the rate of change of its radius;</td>
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<tr>
<td>(b) the rate of change of the surface area.</td>
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<tr>
<td>(2) The displacement ( x ) of a moving particle measured from a fixed point at time ( t ) is given by</td>
<td></td>
<td></td>
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<tr>
<td>( x = a \sin t + b \cos t ).</td>
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<tr>
<td>(a) Find its velocity and acceleration at time ( t ) and describe the motion of the particle.</td>
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<tr>
<td>(b) Show that the velocity at time ( t ) can be expressed as ( \sqrt{a^2 - b^2 - x^2} ).</td>
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<tr>
<td>4.3 Monotonic functions</td>
<td>2</td>
<td>To begin with, teachers may state an intuitively obvious result that</td>
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<td>if ( f(a) &gt; 0 ) then ( f(x) &lt; f(a) ) for values of ( x ) less than ( a ) but sufficiently close to ( a ), and ( f(x) &gt; f(a) ) for values of ( x ) greater than ( a ) but sufficiently close to ( a ).</td>
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<td>From a geometrical point of view, the result can easily lead to the statement that ( f(x) ) is strictly increasing at ( x = a ). (N.B. It is assumed that the function under consideration is continuous and differentiable.) Similar description should be provided for ( f(x) ) strictly decreasing at ( x = a ). Following this, the idea of monotonic increasing may be presented as follows:</td>
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<td>if ( f(x) &gt; 0 ) for every ( x ) of the interval, then ( f(x) ) is a monotonic increasing function throughout the interval.</td>
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<td>Teachers should help the students to derive the following important result:</td>
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<td>if ( f(x) &gt; 0 ) throughout the interval, ((a, b)), ( f(x) ) continuous at ( x = a ) and ( f(a) \geq 0 ), then ( f(x) ) is positive throughout the interval.</td>
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<td>A parallel treatment for monotonic decreasing function is expected and this may be done by the students.</td>
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<tr>
<td>4.4 Maxima and minima</td>
<td>5</td>
<td>This result is of special relevance in proving inequalities like</td>
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<tr>
<td></td>
<td></td>
<td>(i) ((1 + x)^a \leq 1 + ax ) for ( 0 &lt; a &lt; 1 ) and ( x &lt; -1 )</td>
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<td></td>
<td></td>
<td>(ii) ((1 + x)^a \geq 1 + ax ) for ( a &lt; 0 ) or ( a &gt; 1 ) and ( x \geq -1 )</td>
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<td></td>
<td>(iii) ( \sin x + x &gt; \frac{x^3}{3} ) for ( x &gt; 0 )</td>
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<td>The geometrical interpretation of derivative as the gradient of a curve should be explained and emphasized following the introduction of the definition of the gradient of a curve. In this connection, the visualization of a curve being increasing or decreasing can be once again reinforced.</td>
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<td>Consequently upon the mastery of this knowledge, students may then be led to acquire the ability of identifying points of local maximum and local minimum (i.e. the turning points of the curve.) They should be helped to appreciate the conditions for the occurrence of local extrema, like the following version:</td>
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<tr>
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<td>For a function ( f(x) )</td>
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<td></td>
<td>(a) find a such that ( f(a) = 0 ) and</td>
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<td>(b) test the sign of ( f'(a) ) or test for change of sign of ( f(x) ) in a neighbourhood of ( a ).</td>
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<td>Teachers should remind students of the following points:</td>
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<td></td>
<td>(i) Local or relative extrema are not necessarily the global or absolute extrema;</td>
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<td>(ii) turning points may occur at points where the derivatives do not exist; e.g. ( y = x^{2/3} ) and ( y =</td>
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<td></td>
<td>(iii) Stationary points are points whose derivatives are zero;</td>
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<td></td>
<td>(iv) ( f'(a) = 0 ) is NOT sufficient to conclude that at ( x = a ) there is a local extremum;</td>
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<td></td>
<td></td>
<td>e.g. ( x^2 \sin \left( \frac{1}{x} \right) ) and ( x^3 ).</td>
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<td>Examples illustrating the foregoing skills and remarks should be worked out and discussed thoroughly with the students prior to the discussion on point of inflection. The procedures commonly adopted is as follows:</td>
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<td></td>
<td>(a) find a such that ( f'(a) = 0 )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(b) test ( f''(x) ) for change of sign in a neighbourhood of ( a ).</td>
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</tbody>
</table>
Teachers should remind students of the following points:

(i) at points of inflection, the derivative may not be equal to zero;
(ii) $f''(a) = 0$ is not sufficient to conclude the occurrence of an inflection point at $x = a$, e.g. $x^4$.

Hence, diagrams showing the different orientation of the inflectional tangents are very helpful.

As final touching up, teachers may elaborate briefly on the idea of absolute extrema in relation to the domain of the function being extended or shrunken.

Prior to full embarkation on this topic, students should be taught how to determine the vertical, horizontal and oblique asymptotes to a curve whenever they exist. It is recommended that illustrating examples should go with the explanation.

For example, the curve of $y = \frac{x^3}{x^2 - 1}$.

Students should be alerted to look for points of discontinuity where vertical asymptotes are likely to exist. They should also be led to study the behaviour of the function at infinity, viz,

\[ y = x^{\frac{3}{x^2 - 1}} \]

\[ \text{y} = \frac{x}{x^2 - 1} \text{ as y is sufficiently large and so they can realize that} \]

\[ y = x \text{ represents an asymptote.} \]

To accomplish this topic, teachers are advised to help students look for, extract and organize every single bit of glue and information so as to sketch the curve in a more systematic way. The following points are noteworthy:

1. Symmetry about the axes: inspect the equation to detect any symmetry using rules
   - (a) if no odd powers of $y$ appear the curve is symmetrical about the $x$-axis.
   - (b) if no odd powers of $x$ appear the curve is symmetrical about the $y$-axis.

   e.g. \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \) is symmetric about both axes.

2. Limitation on the range of values of $x$ and $y$.
   - e.g. (a) For $y^2 = 4x$, $x$ must be non-negative while all values of $y$ are permissible.
   - (b) For $x^2y^2 = a(x^2 - y^2)$, upon re-writing $y^2 = \frac{a^2x^2}{x^2 + a^2}$, thus all values of $x$ are permissible whereas upon another presentation as $x^2 = \frac{a^2y^2}{a^2 - y^2}$, it is obvious that $|y| < a$. Actually, the curve is included between the asymptotes $y = \pm a$.

3. Intercepts with the axes or any obvious points on the curve.
   - e.g. For $y = \frac{x(x + 2)}{x - 2}$, the curve intercepts the $x$-axis at $-2$ and $0$ and there is no intercept made with the $y$-axis except at the origin.

4. Points of maximum, minimum and inflection.

5. Asymptotes to the curve.

To encompass the various facets, examples should be worked out for students' heeding, however for trigonometric functions, the attention to the period of the curve is desirable. Regarding curve given by parametric equations, no specific rules can be taken heed of and it is advisable to obtain the corresponding Cartesian representation prior to sketching it.

Some typical curves illustrating the above steps should be sketched for students' reference. The following may be considered:

\[ \frac{\text{y}}{\text{indicates the parts deleted}} \]
<table>
<thead>
<tr>
<th>Detailed Content</th>
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<th>Notes on Teaching</th>
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</thead>
<tbody>
<tr>
<td>1. ( y^2 = ax^2 )</td>
<td><img src="image1" alt="Graph" /></td>
<td></td>
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<tr>
<td>2. ( y^2 (a-x) = x^2 )</td>
<td><img src="image2" alt="Graph" /></td>
<td></td>
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<tr>
<td>3. ( y = \frac{1}{x^2 + a^2} )</td>
<td><img src="image3" alt="Graph" /></td>
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</tr>
<tr>
<td>4. ( x^3 - 3axy + y^3 = 0 )</td>
<td><img src="image4" alt="Graph" /></td>
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### Unit B5: Integration

**Objective:**
1. To understand the notion of integral as limit of a sum.
2. To learn some properties of integrals.
3. To understand the Fundamental Theorem of Integral Calculus.
4. To apply the Fundamental Theorem of Integral Calculus in the evaluation of integrals.
5. To learn the methods of integration.
6. To acquire the first notion of improper integral.

#### Detailed Content

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<th>Detailed Content</th>
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<tbody>
<tr>
<td><strong>5.1 The Riemann definition of Integration</strong></td>
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</table>

The theory of the definite integral can be presented in two distinct ways, according as we adopt the geometrical approach or the analytical approach. In the former, the idea of area is presumed, while in the latter the notion of the definite integral as the limit of an algebraic sum without any appeal to geometry is employed. Teachers should determine their choices and sequences of teaching according to the needs of their students. Teachers may start with a function \( f(x) \geq 0 \) for easy understanding and the following simplified version of an intuitive approach is for reference:

Let the function \( f(x) \geq 0 \) in the interval \([a, b]\) and therein let the graph of \( y = f(x) \) be finite and continuous.

![Graph of y = f(x)](image)

\[ y = \frac{x^2}{x^2 - 1} \]
Partition \([a, b]\) into \(n\) subintervals by points \(x_0, x_1, x_2, \ldots, x_n\), such that \(a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b\) and let \(\Delta x_i\) denote \(x_i - x_{i-1}\), and \(z_i\) be an arbitrary point in \([x_{i-1}, x_i]\). The area of the region bounded by the curve \(y = f(x)\), the ordinates \(x = a\) and \(x = b\) and the \(x\)-axis can be approximated by the sum

\[
\sum_{i=1}^{n} f(z_i) \Delta x_i
\]

Moreover, when \(n\) increases and \(\max(\Delta x_i) \to 0\), the value of area can be found and such limit of sum is defined as the definite integral of \(f(x)\) from \(x = a\) to \(x = b\) and it is denoted by

\[
\int_{a}^{b} f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(z_i) \Delta x_i
\]

In the notation,

\(f(x)\) is called the integrand; \(a\) is called the lower limit; \(b\) is called the upper limit and the sum is called the Riemann sum.

**Example 1:**

Consider equal intervals \(\Delta x = \frac{b-a}{n}\) (say) then \(x_0 = a, x_1 = a + h, \ldots, x_n = a + (n-1)h\). Choose \(z_i\) such \(z_i = a + (i-1)h\). As \(\max \Delta x_i = \Delta x = h\),

\[
\int_{a}^{b} e^x \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} e^{z_i} \Delta x_i = \lim_{n \to \infty} \sum_{i=1}^{n} e^{a + (i-1)h} \frac{b-a}{n} = \lim_{n \to \infty} \sum_{i=1}^{n} e^{a} \frac{b-a}{n}
\]

\[
= \lim_{h \to 0} \frac{e^b - e^a}{b-a} \cdot \frac{b-a}{n} \cdot \frac{1}{n} = \frac{e^b - e^a}{b-a}
\]

**Example 2:**

Consider \(n\) intervals such that \(x_0 = a, x_1 = ar, \ldots, x_n = ar^n = b\). When \(n \to \infty\), we have \(b = ar^n \Rightarrow r = \left( \frac{b}{a} \right)^{1/n}\) so that \(r \to 1\), and \(\max \Delta x_i = \Delta x_n = x_n - x_{n-1} = ar^n - ar^{n-1} = a(r^{n-1}) = a(1 - r^{-1}) \to 0\)

Choose \(z_i = x_i = ar^{i-1}\)

\[
\int_{a}^{b} x^m \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} \left( ar^{i-1} \right)^m \left( ar^{i-1} \right) \cdot \frac{b-a}{n} = \lim_{n \to \infty} \sum_{i=1}^{n} a^m \cdot r^{m(i-1)} (r-1)
\]

\[
= \lim_{r \to 1} a^m \cdot \frac{1}{r^m - 1} - 1
\]
5.2 Simple properties of definite integrals

(1) \( \int_a^b k f(x) \, dx = k \int_a^b f(x) \, dx \), \( k \) being a constant

(2) \( \int_a^b f(x) \, dx + \int_a^b g(x) \, dx = \int_a^b [f(x) + g(x)] \, dx \)

(3) \( \int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx \) where \( c \) is any point inside or outside the interval \([a, b]\).

(4) If \( f(x) \geq g(x) \) for all values of \( x \) in \([a, b]\), then \( \int_a^b f(x) \, dx \geq \int_a^b g(x) \, dx \)

(5) If \( |f(x)| < M \) for all values of \( x \) in \([a, b]\), then

\[ \int_a^b |f(x)| \, dx 
\]

In particular, \( a) \) if \( g(x) = |f(x)| \), then

\[ \int_a^b |f(x)| \, dx \leq \int_a^b f(x) \, dx \]

\( b) \) if \( g(x) = M \), \( M \) being constant, then

\[ \int_a^b f(x) \, dx \leq M(b-a) \]

5.3 The Mean Value Theorem for Integrals

Simple and straightforward applications like the following may be discussed with the students:

(1) If \( f(x) \) is positive and monotonic increasing for \( x > 0 \), prove that

\[ f(n-1) \leq \int_n^\infty f(x) \, dx \leq f(n) \]

(2) \[ \int_0^\infty \frac{\sin x}{1+x^2} \, dx \leq \frac{\pi}{4} \]

A simplified version of the theorem is advisable, viz

If \( f(x) \) is continuous on \([a, b]\), then there exists a number \( \xi \) in \((a, b)\) such that

\[ \int_a^b f(x) \, dx = f(\xi)(b-a) \]

The idea conveyed can easily be visualized through the accompanying diagram. Students should find no difficulty to understand the intrinsic meaning of \( f(\xi) (b-a) \) being the area of the rectangle ABCD.
5.4 Fundamental Theorem of Integral Calculus and its application to the evaluation of integrals

5.4 Fundamental Theorem of Integral Calculus and its application to the evaluation of integrals

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If a more formal proof is desirable, it can be furnished by using the properties mentioned in 5.2 together with the properties of continuous function and in particular, the Intermediate Value Theorem.

The First Fundamental Theorem of Integral Calculus, viz

Let \( f(x) \) be continuous on \([a, b]\) and

\[
\int_a^b f(t) \, dt \leq \int_a^b f(x) \, dx.
\]

then

(i) \( F(x) \) is continuous in \([a, b]\)

(iii) \( F(x) \) is differentiable in \([a, b]\) and \( \frac{d}{dx} F(x) = f(x) \)

or the simplified version

if \( f(x) \) is continuous, then the function

\[ F(x) = \int_a^b f(t) \, dt \]

is differentiable and its derivative is equal to the value of the integrand at the upper limit of integration i.e. \( F'(x) = f(x) \). This should be discussed thoroughly with the students and students may be, under the supervision of their teachers, led to prove the theorem using the Mean Value Theorem for Integral Calculus.

(N.B. Teachers should, immediately following this theorem, elaborate on the results follow:

1. The function \( F(x) \) whose derivative is equal to the integrand \( f(x) \) is called a primitive of \( f(x) \).

2. For two such primitives \( F(x) \) and \( G(x) \) of the same integrand, the derivative of \( F(x) - G(x) \) is identically zero, so \( F(x) - G(x) \) is constant.)

Regarding The Second Fundamental Theorem of Integral Calculus, teachers may again assist their students in the derivation. The version that follows is for consideration:

Let \( f(x) \) and \( F(x) \) be continuous in \([a, b]\);

if \( \frac{d}{dx} F(x) = f(x) \) for \( a < x < b \), then for \( a < x < b \),

\[ \int_a^x f(t) \, dt = F(x) - F(a) \]

in particular

\[ \int_a^b f(x) \, dx = F(b) - F(a) \]

Some enlightening examples in evaluating definite integrals by taking it as an infinite sum in the first place and then by finding its primitive as an alternative solution should be worked out so that students' overall understanding on the theorems taught can be strengthened and hence their awareness of the alternative approach in evaluating integrals through the reverse process of differentiation can be promoted. Teachers may start with simpler ones like the following

\[ \int_a^b x^2 \, dx = \frac{b^3}{3} - \frac{a^3}{3} \]

and end up with other interesting applications like

1. By considering \( f(x) = \frac{1}{x} \) in interval \([1, 2]\), the result that \( \frac{1}{n+1} + \frac{1}{n+2} + \ldots \)

\[ \frac{1}{2n} \to \ln 2 \quad \text{as} \quad n \to \infty \]

2. By considering \( f(x) = \frac{1}{1+x^2} \) over \((0, 1)\), one can show that as \( n \to \infty \),

\[ \lim_{n \to \infty} \frac{1}{\binom{2n}{n} \cdot n!} = \frac{\pi}{4} \]

As a continuation, this section is devoted to focus students' attention to the mechanical process of finding primitive as an alternative approach to evaluate definite integrals. The notation \( \int f(x) \, dx \) representing the indefinite integral of \( f(x) \) should be introduced in the sense that

if \( \frac{d}{dx} F(x) = f(x) \) holds, then \( F(x) \) is said to be an indefinite integral of \( f(x) \) and is denoted by \( F(x) = \int f(x) \, dx \).

Teachers should also point out that indefinite integral of \( f(x) \) is not unique and that if \( F(x) \) is an indefinite integral of \( f(x) \), then \( F(x) + c \) where \( c \) is a constant, is another, treating \( \int f(x) \, dx \) as a primitive of \( f(x) \).

Students are expected to be able to apply the following formulae for evaluating indefinite integrals. As a matter of fact, they can be encouraged to derive some or all of them.
### Detailed Content | Time Ratio | Notes on Teaching
--- | --- | ---

1. \( \int x^n \cos x \, dx = \frac{1}{n+1} x^{n+1} \cos x + C \)
2. \( \int e^x \, dx = e^x + C \)
3. \( \int a^x \, dx = \frac{a^x}{\ln a} + C \)
4. \( \int \sin x \, dx = -\cos x + C \)
5. \( \int \cos x \, dx = \sin x + C \)
6. \( \int \sec x \tan x \, dx = \sec x + C \)
7. \( \int \sec x \tan x \, dx = \sec x + C \)
8. \( \int \sec^2 x \, dx = \tan x + C \)
9. \( \int \cos ecx \cot x \, dx = -\cot x + C \)
10. \( \int \cos ecx^2 \, dx = -\cot x + C \)

Teachers may also remind the students of the following properties:

1. \( \int k f(x) \, dx = k \int f(x) \, dx \) where \( k \) is a constant.
2. \( \int [f(x) + g(x)] \, dx = \int f(x) \, dx + \int g(x) \, dx \)

Students should be encouraged to have adequate practices on a sufficient variety of indefinite integrals in order to testify their mastery of the elementary manipulation to facilitate smoother acquisition of the forthcoming techniques.

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### Detailed Content | Time Ratio | Notes on Teaching

#### 5.6 Method of Integration (A) Method of Substitution

- It is suggested that the substitution formula \( \int f(u) \, du = \int f(g(x)) \, g'(x) \, dx \) need not be proved rigorously, however, teachers are advised to start with simpler and obvious ones like:
  1. \( \int \frac{dx}{x^2 + 1} = \arctan x + C \)
  2. \( \int \frac{dx}{\sqrt{x^2 + 1}} = \ln(x + \sqrt{x^2 + 1}) + C \)
  3. \( \int \frac{dx}{e^x + 1} \) (let \( u = e^x \))
  4. \( \int \frac{dx}{\sqrt{e^x + 1}} \) (let \( u = e^x + 1 \))
  5. \( \int \frac{dx}{\sqrt{(x-a)(b-x)}} \) (let \( x = \frac{a+b}{2} + \frac{b-a}{2} \tan \theta \))
  6. \( \int \frac{dx}{x \sqrt{1 + x^2}} \) (let \( x = \tan \theta \))

The following useful results should also be discussed with students with supporting examples for illustration:

1. \( \int_a^b f(x) \, dx = \int_a^b f(a+b-x) \, dx \) and, in particular
   \[ \int_a^b f(x) \, dx = \int_a^b f(a-x) \, dx \]
(2) If \( f(x) = f(a-x) \), then \( \frac{1}{2} \int_0^a f(x) dx = \frac{\pi}{2} \int_0^a f(x) dx \) and in particular
\[
\int_0^a (\sin x) dx = 2 \int_0^a f(x) dx
\]
(3) If \( f(x) \) is periodic with period \( w \) then
\[
\int_0^{\pi w} f(x) dx = \int_0^w f(x) dx
\]
(4) If \( f(x) \) is an even function, then
\[
\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx
\]
(5) If \( f(x) \) is an odd function, then \( \int_{-a}^a f(x) dx = 0 \)
(6) \( \int_0^{\pi/2} f(x) dx = \int_0^w f(x) dx = \int_0^\pi f(x) dx \)

Related examples suggested for consideration are as follows:

(1) \( \int_0^\pi \cos^3 x \sin x + \cos x \) dx
(2) \( \int_0^\pi \sin x + \cos x \) dx
(3) Show that \( \int_0^\pi x^n (a-x)^m dx = \int_0^a x^n (a-x)^m dx \), and hence evaluate
\[
\int_0^a x^3 \frac{1}{2} dx
\]
(4) \( \int_0^\pi x^4 \sin x dx \)
(5) Show that \( \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x} dx = \int_0^{\pi/2} \frac{\sin x}{\cos x + \sin x} dx \) and hence evaluate the integral.
(6) Show that \( \int_0^{\pi/2} \frac{\sin^2 x}{\sin^2 x + \cos^2 x} dx = \pi/4 \).

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| (B) Integration by Parts | 3 | The integration by parts formula \( \int f(x) g'(x) dx = f(x) g(x) - \int g(x) f'(x) dx \) or
\[
\int uv = uv - \int vdu
\]
can readily be proved using the intuitive geometrical approach, like

The diagram suggests an informal geometrical interpretation of the formula:
Area of region A can be represented by \( \int vdu \);
Area of region B by \( \int udv \);
Area of OPQR by \( \int uv \) and hence the formula is readily depicted.
Typical examples for illustration include \( \int x^2 dx \), \( \int x \sin x dx \) and \( \int x^2 dx \)

With the combination of the method of substitution and integration by parts formula, students are able to handle many different kinds of integrals like
(1) \( \int x^2 \cos bx dx \)
(2) \( \int x \tan^{-1} x^2 dx \)
(3) \( \int \frac{x^2}{1-x^2} dx \)
(4) \( \int \sin^{-1} x dx \)

\[\text{Indicates the parts deleted}\]
(C) Reduction Formula

Reduction formula is used to express the integral of the general member of a class of functions in terms of simpler member(s) of the class. The reduction formula is generally obtained by applying the method of integration by parts. It is quite extensively used in the integration of trigonometric functions. Typical examples for consideration are as follows:

1. Let \( I_n = \int x^n \tan^{-1} x \, dx \), show that \( I_n = -\frac{1}{n-1} - I_{n-2} \), \( n \geq 2 \) hence
   
   Evaluate \( I_n \).

2. Let \( I_n = \int \frac{dx}{(x^2 + a^2)^n} \), obtain a reduction formula for \( I_n \) and then evaluate
   
   \( \int_0^\infty \frac{dx}{(x^2 + a^2)^2} \).

3. If \( I_n = \int x^n e^x \, dx \), show that
   
   \( I_n = \frac{1}{n+1} x^n e^x + \frac{1}{2} (n-1) I_{n-2} \) for \( n > 2 \).

(D) Integration by Partial Fractions

Integration of rational algebraic functions may be achieved by splitting the expressions into partial fractions. There are four types of fractions in general:

\[ \frac{f(x)}{ax + b}, \frac{f(x)}{(ax + b)^r}, \frac{f(x)}{(ax^2 + bx + c)^r}, \frac{f(x)}{(ax^2 + bx + c)} \]

Students should be able to handle the first three types without significant difficulty, while for the last type, the application of reduction formula is required. Some examples suggested for discussion are:

1. \( \int_1^4 \frac{x+3}{x+1} \, dx \)

2. \( \int \left( \frac{x}{x^2 - 3x + 2} \right)^2 \, dx \)

5.7 Improper Integrals

The first notion of improper integral is to be introduced and students are expected to be able to recognize improper integral of the first type viz,

\[ \lim_{a \to \infty} \int_a^b f(x) \, dx \] or \( \lim_{b \to -\infty} \int_a^b f(x) \, dx \) which may be simply denoted by \( \int_a^b f(x) \, dx \) or \( \int_{-\infty}^\infty f(x) \, dx \)

and improper integral of the second type, viz.

\[ \lim_{b \to a^+} \int_a^b f(x) \, dx \] when \( \lim_{x \to a^+} f(x) = \infty \)

\[ \lim_{b \to \infty} \int_a^b f(x) \, dx \] when \( \lim_{x \to \infty} f(x) = \infty \).

Typical examples of the first type:

1. \( \int_0^\infty \frac{dx}{1+x} \)

2. \( \int_0^\infty \frac{dx}{x^2} \)

The integrals are put forth the example \( \int_0^\infty \frac{dx}{x} \) and pinpoint that this is not an improper integral as the limit does not exist.
Unit B6: Application of Integration

Objective:
1. To learn the application of definite integration in the evaluation of plane area, arc length, volume of solid of revolution, and area of surface of revolution.
2. To apply definite integration to the evaluation of limit of sum.

Detailed Content | Time Ratio | Notes on Teaching
--- | --- | ---
6.1 Plane area | 5 | As a sequel to the definition of definite integral, the area bounded by a curve \( y = f(x) \), the ordinates \( x = a \) and \( x = b \), and the x-axis can be evaluated in the following ways depending on the nature of the function (being above or below the x-axis):

- **case (i)** when \( y = f(x) \) is continuous and non-negative in \([a, b]\), the area so bounded is given by \( \int_{a}^{b} f(x) \, dx \)
- **case (ii)** when \( f(x) \) is continuous and non-positive in \([a, b]\), the area is given by \(- \int_{a}^{b} f(x) \, dx\)

As indicated in the diagrams:
- **case (i)**
- **case (ii)**

\( 
\begin{align*}
\text{case (i)} & & \text{case (ii)} \\
0 & \quad a & b & \quad 0 \\
& \quad & \text{y = f(x)} & \quad \\
& \quad & \text{y = f(x)} & \\
& \quad & \text{y = f(x)} & \\
x & \quad & \text{y = f(x)} & \\
0 & \quad & \text{y = f(x)} & \\
& \quad & \text{y = f(x)} & \\
& \quad & \text{y = f(x)} & \\
\end{align*} 
\)
case (iii) when $f(x)$ is continuous and has positive and negative values in $[a, b]$, then the area bounded can be found by following the approach shown in the simplified example that follows.

$$
\begin{align*}
\text{Area is given by } & \int_{a}^{b} f(x) \, dx - \int_{a}^{b} g(x) \, dx.
\end{align*}
$$

Students should be reminded of the minus sign for area enclosed below the $x$-axis thus they should be encouraged to have a rough sketch of the function so as to obtain a clearer picture.

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Teachers should also elaborate on the cases where the area enclosed was made with the $y$-axis like the following diagram.

$$
\begin{align*}
\text{Area given by } & - \int_{c}^{d} xy \, dy.
\end{align*}
$$

For area bounded by two curves, the following approach together with other variations which are illustrated diagrammatically should be discussed thoroughly with students with adequate exemplification.

$$
\begin{align*}
\text{Area } & = \int_{a}^{b} (f(x) - g(x)) \, dx
\end{align*}
$$

indicates the parts deleted
When the polar equation of the curve is given say, \( r = f(\theta) \), then the area bounded by the curve and between the two radii is given by \( \frac{1}{2} \int_{\alpha}^{\beta} r^2 \, d\theta \).

6.2 Arc length

The length of arc of the curve \( y = f(x) \) between two points on the curve at \( x = a \) and \( x = b \) is given by

\[
\int_{a}^{b} \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx
\]

If the curve is given in parametric form \( x = x(t); y = y(t) \) then the arc length of the curve from \( t = t_1 \) to \( t = t_2 \) is given by

\[
\int_{t_1}^{t_2} \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2} \, dt
\]

If the curve is given in polar form \( r = f(\theta) \), then the arc length of the curve from \( \theta = \alpha \) to \( \theta = \beta \) is given by

\[
\int_{\alpha}^{\beta} \sqrt{r^2 + \left( \frac{dr}{d\theta} \right)^2} \, d\theta
\]
The following suggested examples are for discussion and illustration:

1. Show that the perimeter of the closed curve (asteroid) \( x = a \cos^3 \theta; \ y = a \sin^3 \theta \) is \( 8a \).
   (Symmetry of the curve about the axes helps)

2. Show that the length of the circumference of the cardioid \( r = a(1 + \cos \theta) \) is \( 4a \).

3. Find the length of the arc of the curve \( x^2 = 8y^2 \) from \( x = 1 \) to \( x = 3 \).

(N.B. In choosing curves for illustration, teachers should be well aware of the case of an ellipse and, if deemed desirable, may lead a brief discussion with students so as to broaden their perspective on other branches of mathematics studies. A brief account in this respect is suggested for reference as follows:

For the ellipse \( x = a \sin \theta; \ y = b \cos \theta \) \( b^2 \sin^2 \theta = a^2 (1 - \epsilon^2 \sin^2 \theta) \) where \( \epsilon = \frac{b}{a} \) (commonly known as the eccentricity).

The arc is measured from an extremity of the minor axis is given by
\[ a \int_0^\theta \sqrt{1 - \epsilon^2 \sin^2 \theta} \, d\theta. \] This integral cannot be expressed in terms of elementary functions in a finite form. It is called an elliptic integral of the second kind denoted by \( E(\epsilon, \phi) \). For the sake of completeness, teachers may also introduce the elliptic integral of the first kind, viz
\[ \int_0^\phi \frac{d\phi}{\sqrt{1 - \epsilon^2 \sin^2 \phi}} \] which is denoted by \( F(\epsilon, \phi) \).)

It is desirable to have some preliminary discussion with the students on the meaning and formation of solids of revolution while the term axis of revolution should also be introduced so that students may be able to identify solids of revolution and visualize the solids formed when certain segment of curve or region is revolving about certain axis. Teachers may then touch upon the two common methods in finding volume of revolution, viz,

(1) The Disc Method

Teachers should emphasize on the expression of the volume element \( dV \) or \( dV \) that \( dV = \pi y^2 \, dx \) which is the volume of the disc. The volume of the solid is given by
\[ \pi \int_a^b y^2 \, dx. \]

Teachers are also advised to elaborate a bit more on \( \pi \int_a^b x^2 \, dy \) which is the case when the curve is revolving about the \( y \)-axis.

(2) The Shell Method
In this case the volume element is $2\pi xydx$ and the volume is given by $2\pi \int_a^b xydx$.

There are cases in which the solid is formed by revolving about certain lines other than the axes or formed by revolving a region bounded by two curves. Thus teachers should elaborate on these cases with the so-called formulae like $\pi \int_a^b [f(x)]^2 - [g(x)]^2 \, dx$ or $\pi \int_a^b [(f(y))^2 - (g(y))^2] \, dy$ derived for students reference. Adequate illustration is highly recommended. The following are some for reference:

1. Show that the volume generated by rotating the ellipse $x = a\cos\theta, y = b\sin\theta$ about the x-axis is $\frac{4}{3}\pi ab^2$.
   (N.B. Teachers may request the students to deduce the volume of a sphere of radius r from the given result.)

2. Find the volume of the solid formed by revolving about the line $x = 2$ the region which is bounded by the curve $y = x^2$, the line $x = 2$ and the y-axis.
   (N.B. Teachers are advised to solve this problem using the disc approach as well as the shell approach.)

### 6.4 Area of Surface of Revolution

The surface area generated by revolving about the x-axis the arc of the curve $y = f(x)$ between $x = a$ and $x = b$ is given by $2\pi \int_a^b yds$ where $ds$ is the element arc length, and so the formula usually appears as

$$2\pi \int_a^b y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

If the said arc length is revolved about the y-axis, the surface area is given by

$$2\pi \int_c^d xds \quad \text{or} \quad 2\pi \int_c^d x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy$$

where $c$ and $d$ are respectively the ordinates of the end points of the arc.

### 6.5 Limit of Sum

In teaching this interesting application of definite integral, it is advisable to start with some simple and obvious series like $\frac{1}{2^2} + \frac{2}{3^2} + \ldots + \frac{n-1}{n^2} + \frac{n}{n^2}$, and students should be provided with adequate hints so that they manage to associate the limit of sum of the series with the limit of sum leading to the relevant definite integral in a suitable interval and with pertinent partition and most important of all with the appropriate integrand, viz

$$f(x) = x \ln [0, 1] \text{ with partition } 0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1.$$
It would not be a difficult task for students to establish the result that
\[
\lim_{n \to \infty} \left( \frac{1}{n} + \frac{1}{n^2} + \ldots + \frac{1}{n^n} \right) = \lim_{n \to \infty} \frac{n}{n!} \cdot \frac{1}{n}
\]
\[
= \int_0^1 \frac{dx}{\sqrt{1 + x}} = \frac{\pi}{4}
\]

Other examples of great mathematical insight like

1. \[
\lim_{n \to \infty} \left( \frac{1}{n} + \frac{1}{n + 1} + \ldots + \frac{1}{2n - 1} \right) = \int_0^1 \frac{dx}{1 + x^2} = \ln 2
\]

2. \[
\lim_{n \to \infty} \left( \frac{1}{n^3} + \frac{1}{n^2 + 1} + \ldots + \frac{n}{n^2 + (n - 1)^2} \right) = \int_0^1 \frac{dx}{\sqrt{1 + x}} = \frac{\pi}{6}
\]

3. \[
\lim_{n \to \infty} \left( \frac{1}{\sqrt{n^2 + 1}} + \frac{1}{\sqrt{n(n+1)}} + \ldots + \frac{1}{\sqrt{n(2n - 1)}} \right) \]
\[
= \lim_{n \to \infty} \left( \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{1}} + \ldots + \frac{1}{\sqrt{n - 1}} \right)
\]
\[
= \int_0^1 \frac{dx}{\sqrt{1 + x}} = 2\sqrt{2} - 1
\]

may be provided to further their understanding and manipulative technique. The following example is worth discussing as it brings exhilarating result:

To find \( \lim_{n \to \infty} \frac{\sqrt{n!}}{n} \) by transforming it into

\[\ln y = \lim_{n \to \infty} \sum_{k=1}^{\infty} \ln \left( \frac{1}{k} \right) \]
\[y = \frac{1}{\ln n} \]
\[= \frac{1}{\ln y} \]
\[\ln x \ dx = -1 \]

thus

obtain \( y = e^{-1} \) (N.B. Teachers may relate this part to Unit B5 in which the idea of Riemann sum expressed as a limit of sum of a series is touched upon.)

However, students should be reminded that not all limits of sum can be dealt with using this approach, the harmonic series, \( \sum \frac{1}{n} \), is an example.
Unit B7: Analytical Geometry

Objective: (1) To learn polar coordinates as another system other than the rectangular coordinate system.
(2) To learn the conic sections.
(3) To study locus problems algebraically.
(4) To solve related problems.

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| 7.1 Basic knowledge in coordinate geometry            | 5          | Besides the knowledge in their secondary mathematics, students should acquire the following knowledge before they go on to the other topics in this unit:

   (1) external point of division:
   \[\begin{array}{c|c}
   x & y \\
   \hline
   x_1 & y_1 \\
   x_2 & y_2 \\
   \vdots & \vdots \\
   x_n & y_n \\
   x_t & y_t \\
   \end{array}\]

   (2) area of rectilinear figure using \(\frac{1}{2} \times x \times y\):
   \[\begin{align*}
   x_1 & y_1 \\
   x_2 & y_2 \\
   \vdots & \vdots \\
   x_n & y_n \\
   x_t & y_t \\
   \end{align*}\]

   (3) angle between two lines using \(\tan \theta = \frac{m_1 - m_2}{1 + m_1 m_2}\):

   (4) the normal form of a straight line;
   (5) angle bisectors of two straight lines;
   (6) family of straight lines and
   (7) family of circles.

   Students should be able to make conversions between polar and rectangular coordinate systems. They should know how to change the equation of a curve in polar form into the corresponding rectangular form and vice versa.

   Polar to rectangular:
   \[
   \begin{align*}
   x &= r \cos \theta \\
   y &= r \sin \theta
   \end{align*}
   \]

   Rectangular to polar:
   \[
   r = \sqrt{x^2 + y^2} \quad \text{and} \quad \tan \theta = \frac{y}{x}
   \]
   where \(x = 0\) and \(\theta\) is determined by the quadrant in which \((x, y)\) lies.

   Students should be able to plot curves with their polar equations given. They are the fundamentals to the topic "Applications of Integration". The following are some suggested simple curves in their polar forms that the students should be able to sketch:

   (1) the straight line: \(r = k\) where \(k\) is a positive constant; \(r \cos \theta = a\) (vertical line);
   (2) the circle: \(r = k\) where \(k\) is a positive constant; \(r = \sin \theta\),
   (3) the parabola: \(r(1 + \cos \theta) = k\) where \(k\) is a positive constant
   (4) the cardioid: \(r = \sin (1 - \cos \theta)\) where \(k\) is a positive constant;
   (5) the rose curve: \(r = a \sin n\theta\)

   (6) the spiral: \(r = \theta\)

| 7.2 Sketching of curves in the polar coordinate system | 4          | Students should be able to distinguish the conic sections in the standard position, namely:

   - \(y^2 = 4ax\) (parabola)
   - \(\frac{x^2}{a^2} = \frac{y^2}{b^2} = 1\) (ellipse)
   - \(\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\) (hyperbola)
   - \(xy = c^2\) (rectangular hyperbola)

| 7.3 Conic sections in rectangular coordinate system   | 7          | Students should be able to distinguish the conic sections in the standard position, namely:

   - \(y^2 = 4ax\) (parabola)
   - \(\frac{x^2}{a^2} = \frac{y^2}{b^2} = 1\) (ellipse)
   - \(\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\) (hyperbola)
   - \(xy = c^2\) (rectangular hyperbola)
The parametric representation of the conic sections should also be taught, namely:

\[
\begin{align*}
&y = r \sin \theta \quad : \quad x = r \cos \theta \\
&y = r \tan \theta \quad : \quad x = r \sec \theta \\
&y = \frac{x}{t} \quad : \quad x = ct \\
&y = 2at \quad : \quad x = at^2
\end{align*}
\]

(circle)
(ellipse)
(hyperbola)
(rectangular hyperbola)
(parabola)

The knowledge of asymptotes of a hyperbola is expected. The knowledge of the properties of conic sections such as eccentricity, focus and directrix may be taught but need not be emphasized.

The knowledge of using various methods to find the equations of tanget to a circle is expected. For the simple circle \(x^2 + y^2 = a^2\), the equation of tangent to the circle at \((x_1, y_1)\) a point on the circle, is given by \(xx_1 + yy_1 = a^2\) and the normal by \(xx_1 - yy_1 = 0\). The derivation of these basic results should be provided to lead student's thinking and examples like the following should also be worked out with due emphasis on the underlying methodology:

To find the equation of tangent at \((x_1, y_1)\) on the circle \(x^2 + y^2 + 2gx + 2fy + c = 0\):

(i) using the property that the tangent is always perpendicular to the radius of the circle;

(ii) by letting the equation of tangent be \(y = mx + k\) and using the fact that the simultaneous equations

\[
\begin{align*}
&y = mx + k \\
&x^2 + y^2 + 2gx + 2fy + c = 0
\end{align*}
\]

have equal roots.

It should be noted that the method in (ii) can be applied to the case where \((x_1, y_1)\) is not on the circle. Hence the result that the equation of the tangent to the circle \(x^2 + y^2 + 2gx + 2fy + c = 0\) at \((x_1, y_1)\) on the circle given by \(xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0\) can be obtained and upon generalization, the following results can be obtained:

(a) Tangent to the parabola \(y^2 = 4ax\) at \((x_1, y_1)\) on the curve is \(yy_1 = 2a(x + x_1)\)

(b) Tangent to the ellipse \(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\) at \((x_1, y_1)\) on the curve is \(\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1\)

(c) Tangent to the hyperbola \(\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\) at \((x_1, y_1)\) on the curve is \(\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1\)

(d) Tangent to the rectangular hyperbola \(xy = c^2\) at \((x_1, y_1)\) on the curve is \(xy_1 + x_1y = 2c^2\)

Once the equation of tangent is obtained, students should have no difficulty to obtain the equation of the normal.

Furthermore, the corresponding results when the conic sections concern are presented in parametric form should also be discussed. Teachers may ask the students to do the derivation for themselves:

(a) For the parabola \(\frac{x^2}{a^2} = \frac{y}{t}\), the tangent is \(y = \frac{x}{t} \quad + at\)

(b) For the ellipse \(\frac{x^2}{a} \quad \cos \theta \quad + \quad \frac{y}{b}\) \quad \sin \theta \quad = \quad 1\)

(c) For the hyperbola \(\frac{x}{a} \quad \sec \theta \quad - \quad \frac{y}{b}\) \quad \tan \theta \quad = \quad 1\)

(d) For the hyperbola \(\frac{x}{c} \quad = \quad \frac{y}{t}\), the tangent is \(x + t^2y = 2ct\).

To consolidate students' mastery of the concept as well as the manipulative technique, some basic ideas concerning the chord of contact should be discussed.

Cases in which a certain set of points satisfying certain constraints and can be represented by equations in the rectangular coordinate system should be studied, e.g. (1) The locus of a movable point with fixed distance from a fixed point is a circle.

(2) The locus of a movable point which is equidistant from a fixed point and a fixed line is a parabola.

(3) The locus of a point on the rim of a circle when the circle is rolled on a straight line represent a cycloid.
<table>
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<th>Detailed Content</th>
<th>Time Ratio</th>
<th>Notes on Teaching</th>
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<tr>
<td>7.8 Tangents and normals of plane curves</td>
<td>4</td>
<td>After students have learnt differential calculus, they should be able to apply differentiation to find the equations of tangent and normal of a plane curve in rectangular coordinate plane. Using differentiation formulae and the chain rule, students can find the equations of tangents and normals of curves whose functions are implicitly defined or in parametric form. In this connection, teachers are advised to determine the teaching sequence of this unit, as a prologue to the harder application of differential calculus or as an epilogue of the basic application of differential calculus, according to the ability of the students.</td>
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## Resources for Learning and Teaching of AL Pure Mathematics

### I. Reference Books

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<tr>
<th>Title</th>
<th>Author</th>
<th>Publisher</th>
<th>Year</th>
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<tr>
<td>A Course in Pure Mathematics</td>
<td>Margaret M. Gow</td>
<td>Hodder</td>
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<td>A Course of Pure Mathematics</td>
<td>G.H. Hardy</td>
<td>Cambridge Uni. Press</td>
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<td>Advanced Level Pure Mathematics</td>
<td>S.L. Green</td>
<td>UTP</td>
<td>1989</td>
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<td>Advanced Level Pure Mathematics</td>
<td>C. J. Tranter</td>
<td>The English Universities Press Ltd.</td>
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<tr>
<td>Elementary Calculus</td>
<td>F. Bowman</td>
<td>Longman</td>
<td>1977</td>
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<tr>
<td>Further Pure Mathematics</td>
<td>L. Bostock, S. Chandler &amp; C. Rourke</td>
<td>Trans-Atlantic Publications</td>
<td>1994</td>
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<td>Improving Mathematics Teaching with DERIVE</td>
<td>B. Kutzler</td>
<td>Chartwell-Bratt</td>
<td>1996</td>
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<td>Introduction to Elementary Calculus</td>
<td>S. R. Hsieh</td>
<td>Luen Shing Printing Co.</td>
<td>1991</td>
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<td>Techniques of Mathematical Analysis</td>
<td>C. J. Tranter</td>
<td>Hodder</td>
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<tr>
<td>The Tutorial Algebra (Vol. 1 &amp; 2)</td>
<td>Briggs &amp; Bryan</td>
<td>University Tutorial Press</td>
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<td>幼獣數學大辭典 (上、下)</td>
<td>數學教研室</td>
<td>幼獣文化事業</td>
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<td>高中三角學</td>
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<td>高等數學</td>
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<td>湖南科學技術</td>
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<td>基礎微積分</td>
<td>邵之泉、華青等</td>
<td>知識</td>
<td>1987</td>
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<td>微分、積分(上下合併)</td>
<td>嚴水巖譯</td>
<td>晨光</td>
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<td>微積分探原</td>
<td>吳英格譯</td>
<td>徐氏基金會</td>
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<td>數學分析 (上、下)</td>
<td>復旦大學數學系</td>
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<td>數學分析習題集</td>
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<td>數學和數學家的故事 (1-7)</td>
<td>李學數</td>
<td>廣角鏡</td>
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<td>數學歷史典故</td>
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<td>數學歸納法</td>
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<td>標準高等代數學 (上、下)</td>
<td>陳明哲</td>
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<td>標準解析幾何</td>
<td>陳明哲</td>
<td>中央書局</td>
<td>1976</td>
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</tbody>
</table>
II. Popular Software Packages

1. Cabri Geometry (Texas Instruments)
2. Calculus 1.0 (Ashay Dharwadker)
3. Derive (Texas Instruments)
4. FX Draw (Efofex Software)
5. Geometer's Sketchpad (Key Curriculum Press)
6. Graphmatica (KSoft Inc.)
7. Journey Through Calculus (Brooks/Cole Group)
8. Maple (Waterloo Maple)
9. Mathematica (Wolfram Research)
10. Microsoft Excel (Microsoft Corporation)
11. Scientific Notebook (TCI Software Research)
12. Winplot (Peanut Software)